HOMEWORK 1

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Abstract. Please send me an email if you find mistakes. Thanks.

1. P 5. #1.1

Proof. We prove it by math induction. For $n = 1$, both sides equal to 1.

Suppose the claim is true for $n \in \mathbb{N}$. We prove that it is true for $n + 1$. We consider

$$1 + 2 + \cdots + n^2 + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

by the induction hypothesis. We simplify it further to obtain

$$= \frac{1}{6}(n+1)\left(2n^2 + n + 6n + 6\right) = \frac{1}{6}(n+1)(n+2)(2n+3) = \frac{1}{6}(n+1)[(n+1)+1][2(n+1)+1].$$

Thus we prove that

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

□

2. P 5. #1.7

Proof. We prove it by math induction. For $n = 1$, it is $7^n - 6n - 1 = 0$, which is divided by 36.

Suppose the claim is true for $n \in \mathbb{N}$. That is to say,

$$7^n - 6n - 1 = 36k,$$

for some $k \in \mathbb{N}$. We consider

$$7^{n+1} - 6(n+1) - 1 = 7 \times 7^n - 6n - 7 = 7(6n+1+36k) - 6n - 7 = 36n + 36 \times 7k$$

which is divisible by 36. Therefore we prove the claim. □

3. P5. # 1.12

Proof. **Part (a).** We skip it.

Part (b). By the formula,

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!},$$

Then

$$\begin{aligned}
&\left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k-1 \end{array} \right) \\
&= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!} = \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k} + \frac{1}{n-k+1}\right) \\
&= \frac{n!}{(k-1)!(n-k)!}\frac{n+1}{k(n-k+1)} \\
&= \frac{(n+1)!}{k!(n-k+1)} \\
&= \left(\begin{array}{c} n+1 \\ k \end{array} \right).
\end{aligned}$$

Part (c). The claim is true for $n = 1$. Suppose that it is true for n. Then for $n + 1$,

$$\begin{aligned}
&(a+b)^{n+1} = (a+b)^n(a+b) \\
&= \left(\left(\begin{array}{c} n \\ 0 \end{array} \right) a^n + \left(\begin{array}{c} n \\ 1 \end{array} \right) a^{n-1}b + \cdots + \left(\begin{array}{c} n \\ n-1 \end{array} \right) ab^{n-1} + \left(\begin{array}{c} n \\ n \end{array} \right) b^n\right)(a+b) \\
&= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k}b^k(a+b) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n+1-k}b^k + \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k}b^{k+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \left(\left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k-1 \end{array} \right)\right) a^{n+1-k}b^k + b^{n+1} \\
&= \left(\begin{array}{c} n+1 \\ 0 \end{array} \right) a^{n+1} + \sum_{k=1}^{n} \left(\begin{array}{c} n+1 \\ k \end{array} \right) a^{n+1-k}b^k + \left(\begin{array}{c} n+1 \\ n+1 \end{array} \right) b^{n+1} \\
&= \sum_{k=0}^{n+1} \left(\begin{array}{c} n+1 \\ k \end{array} \right) a^{(n+1)-k}b^k.
\end{aligned}$$

This proves the claim. □

4. P.13. # 2.3

Proof. Let $r = \sqrt{2+\sqrt{2}}$. Then

$$r^2 - 2 = \sqrt{2}.$$

Then

$$(r^2 - 2)^2 = (\sqrt{2})^2 = 2.$$

We simplify it to obtain

$$r^4 - 4r^2 + 2 = 0.$$

If r is a rational number, then r divides 2 and r is an integer. So there are 4 possibilities of 2, ± 1 and ± 2. However $r = \sqrt{2+\sqrt{2}}$ is not one of them. This is a contradiction. It proves that r is not a rational number. $\square$

5. P.13. # 2.7.

Proof. We complete squares to prove that these are rational numbers. Since $4 + 2\sqrt{3} = (\sqrt{3}+1)^2$,

$$\sqrt{4+2\sqrt{3}} - \sqrt{3} = (1+\sqrt{3}) - \sqrt{3} = 1.$$

Similarly one can also prove that $\sqrt{6+4\sqrt{2}} - \sqrt{2}$ is a rational number. $\square$

6. P.19. # 3.1.

Proof. **(a).**For $\mathbb{N}$, A3, A4, M4 fail.

(b). For $\mathbb{Z}$, M4 fails.

$\square$

7. P.19,# 3.5.

Proof. **(a).** Since $|b| \leq a$, we prove that $-a \leq b \leq a$.

If $b \geq 0$, $0 \leq a$, $b \leq a$. On the other hand, if $b < 0$, $|b| = -b$,

$$-b \leq a.$$

Hence $-a \leq b$. Together we have

$$-a \leq b \leq a.$$

To prove the converse direction, If $b \geq 0$,

$$|b| = b \leq a.$$

If $b < 0$, since $-a \leq b$,

$$|b| = -b \leq a.$$

This proves that $-a \leq b \leq a$. □

8. P. 19. # 3.8

Proof. We prove it by contradiction. If $a > b$, let $\epsilon = \frac{a-b}{2}$. Then

$$b + \epsilon = b + \frac{a-b}{2} = \frac{a+b}{2}.$$

This is a number strictly larger than b since $\epsilon > 0$. On other hand side,

$$\frac{a+b}{2} = a - \epsilon$$

which is strictly less than a. A contradiction. Therefore $a \leq b$. □

9. P26. # 4.3 & # 4.4

Proof. For these two problems, we give several examples to show how we achieve the supremum and the infimum.

For # 4.3, we take (a), (e), (k) and (w) as examples. For (a), $\sup = 1$. For (e), $\sup = 1$. For (k), this set is not bounded and so there is no supremum. For (w), since sin is a periodic function, there are only 3 values for $\sin \frac{n\pi}{3}$ for $n \in \mathbb{N}$:

$$0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}.$$

Therefore the supremum is $\frac{\sqrt{3}}{2}$.

For # 4.4, we take (c), (i) and (n) as examples. For (c), $\inf = 2$. For (i), $\inf = 0$. For (n), $\inf = -\sqrt{2}$. □

10. P27. # 4.5

Proof. Firstly for any $a \in S$, $a \leq \max S$. So $\max S$ is an upper bound. Secondly for any upper bound α of S, since $\max S \in S$, $\alpha \geq \max S$. Then by the definition of supremum, we see that $\max S = \sup S$. □

11. P27. # 4.9

Proof. Consider $-S = \{-x : x \in S\}$. Then $-S \neq \emptyset$. By hypothesis, if S is bounded below, then $-S$ is bounded above. So $\sup(-S)$ exists, which we denote by a. For $x \in S$,

$$-x \leq a, \Rightarrow x \geq -a.$$

For any lower bound β of S, $-\beta$ is an upper bound of $-S$. Thus we see that

$$a = \sup(-S) \leq -\beta, \Rightarrow -a \geq \beta.$$

Thus $-a$ is the infimum of S, $\inf S = -a = -\sup(-S)$. □

12. P27. # 4.10

Proof. By Archemedian's property, since $a > 0$, there exists $n_1 \in \mathbb{N}$ such that $n_1 a > 1$. Hence $a > \frac{1}{n_1}$. On the other hand, for $1 > 0$, there exists $n_2 \in \mathbb{N}$ such that $n_2 \times 1 > a$. Therefore we take $n = \max\{n_1, n_2\}$ and obtain

$$\frac{1}{n} < a < n.$$

□

13. P27. # 4.12

Proof. Firstly there exists a rational number $r \in \mathbb{Q}$ such that $a < r < b$ by the density property of rational numbers in the real numbers. On the other hand, since $\sqrt{2} > 0$, there exists $n \in \mathbb{N}$ such that

$$n(b - r) > \sqrt{2},$$

which implies that

$$b - r > \frac{\sqrt{2}}{n}.$$

We consider $x = r + \frac{\sqrt{2}}{n}$ that is irrational. Then $x < r + (b - r) = b$.

$$a < x < b.$$

□

14. P27. # 4.12

Proof. **(a).**Since A and B are bounded sets, $\sup A$ and $\sup B$ exist; $A+B$ is also a bounded set, therefore $\sup(A+B)$ exists.

For any $a \in\in (A+B)$, $a = x+y$ for $x \in A$ and $y \in B$. Therefore

$$a = x + y \leq \sup A + \sup B,$$

which implies that,

$$\sup(A+B) \leq \sup A + \sup B.$$

On the other hand, for any $x \in A$ and $y \in B$, $x+y \in A+B$.

$$x + y \leq \sup(A+B).$$

Fix y, the above implies that

$$x \leq \sup(A+B) - y.$$

Therefore

$$\sup A \leq \sup(A+B) - y.$$

To continue, we rewrite it as follows,

$$y \leq \sup(A+B) - \sup A.$$

which implies,

$$\sup A + \sup B \leq \sup(A+B).$$

Therefore

$$\sup A + \sup B = \sup(A+B).$$

(b). This follows from part (a) and Ex. 4.9. □

15. P28. # 4.16

Proof. This follows from density of rational numbers in $\mathbb{R}$. □

16. P30. # 5.4

Proof. By Ex. 4.9, we just need to prove the case where $\inf S = -\infty$. This is the case where S is not bounded below. So $-S$ is not bounded above. So

$$\sup(-S) = +\infty.$$

Hence

$$\inf S = -\sup(-S).$$

□

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