HOMEWORK 2

SHUANGLIN SHAO

ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P38. #7.3

Proof. The sequences in **(b), (d), (f), (j),(p),(r), (t)** converge; their limits are $1, 1, 1, \frac{7}{2}, 2, 1$ and 0, respectively.

The limit in **(h)** diverges because it is not bounded.

The limit in **(l)** diverges because it is a sequence consisting of $0, 1, -1$; the limit in **(n)** diverges because it is a sequence consisting of $0, \frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{2}$. You can refer to Example 4 in Section 8 for a discussion. □

P38. # 7.4

Proof. **(a).** $x_n = \frac{1}{n\sqrt{2}}$; Then

$$\lim_{n\to\infty} x_n = 0.$$

(b). $r_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Then

$$\lim_{n\to\infty} r_n = \frac{\pi^2}{6}.$$

□

2. P44. # 8.2

Proof. **(b).** $\lim_{n\to\infty} b_n = \frac{7}{3}$. For any $\epsilon > 0$, we need to find $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\left| \frac{7n-19}{3n+7} - \frac{7}{3} \right| < \epsilon. \tag{1}$$

In order for (3) to hold,

$$n > \frac{106}{9\epsilon} - \frac{7}{3}.$$

So we take $N \in \mathbb{N}$ and $N > \frac{106}{9\epsilon} - \frac{7}{3}$. To conclude, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\left| \frac{7n-19}{3n+7} - \frac{7}{3} \right| < \epsilon.$$

Similarly in **(d)**,

$$\lim_{n\to\infty} d_n = \frac{2}{5}.$$

□

3. P44. # 8.4

Proof. $\lim_{n\to\infty} s_n = 0$: for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$|s_n - 0| \leq \frac{\epsilon}{M+1}.$$

Then

$$|s_n t_n - 0| \leq |t_n||s_n| \leq M\frac{\epsilon}{M+1} < \epsilon.$$

This proves that $\lim_{n\to\infty} s_n t_n = 0$. □

4. P44. # 8.5(a)(b).

Proof. **(a).** This is proven in class. Please refer to the class notes.

(b). $\lim_{n\to\infty} t_n = 0$: for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|t_n - 0| = t_n < \epsilon.$$

Since $|s_n| \leq t_n$,

$$|s_n - 0| < \epsilon.$$

Therefore

$$\lim_{n\to\infty} s_n = 0.$$

□

5. P44. # 8.6

Proof. **(a) (b).**This is proven in class. Please refer to the class notes. □

6. P45. # 8.8

Proof. **(a).**

$$0 \le \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n} \le \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$,

$$\lim_{n\to\infty} \sqrt{n^2+1} - n = 0.$$

(b). For any $\epsilon > 0$, we need to find $N \in \mathbb{N}$ such that for any $n \ge N$,

$$|(\sqrt{n^2+n} - n) - \frac{1}{2}| < \epsilon. \tag{2}$$

We know that

$$|(\sqrt{n^2+n} - n) - \frac{1}{2}| = \left| \frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2} \right| = \frac{n}{2(\sqrt{n^2+n}+n)^2} \le \frac{1}{2n}.$$

In order for (2) to hold, it suffices that

$$\frac{1}{2n} < \epsilon, \text{i.e.}, \ n > \frac{1}{2\epsilon}.$$

We can take $N \in \mathbb{N}$ and $N > \frac{1}{2\epsilon.}$.

The proof in **(c)** is similar. □

7. P54. # 9.4

Proof. **(a).**$s_1 = 1, s_2 = \sqrt{2}, s_3 = \sqrt{\sqrt{2}+1}, s_4 = \sqrt{\sqrt{\sqrt{2}+1}}.$

(b). Let $A = \lim_{n\to\infty} s_n$. Then taking limits on both sides of $s_{n+1} = \sqrt{s_n+1}$.

$$A = \sqrt{A+1}.$$

Therefore

$$A^2 - A - 1 = 0.$$

So $A = \frac{1\pm\sqrt{5}}{2}$. Since $A \ge 0$, we see that

$$A = \frac{1+\sqrt{5}}{2}.$$

□

8. P54. # 9.5

Proof. Let $A = \lim_{n\to\infty} t_n$. Obviously by math induction $t_n \geq 0$; then $t_{n+1} \geq \frac{2\sqrt{2}t_n}{2t_n} \geq \sqrt{2}$. Taking limits on both sides of $t_{n+1} = \frac{t_n^2+2}{2t_n}$, we see that

$$A = \frac{A^2+2}{2A}.$$

This proves that

$$A = \sqrt{2}.$$

□

9. P55. #9.9

Proof. **(a).** If $\lim_{n\to\infty} s_n = \infty$, for any $M > 0$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$,

$$s_n \geq M.$$

Taking $n \geq \max\{N_0, N_1\}$ and then we see that

$$t_n \geq M.$$

This proves that $\lim_{n\to\infty} t_n = \infty$.

The claim in **(b)** is proven similarly.

(c). For the claim in **(c)**, we prove it by contradiction. Suppose that $\lim_{n\to\infty} s_n > \lim_{n\to\infty} t_n$. If $\lim_{n\to\infty} s_n = +\infty$, then by **(a)**, we see that $\lim_{n\to\infty} t_n = \infty$, A contradiction. if $\lim_{n\to\infty} t_n = -\infty$, then $\lim_{n\to\infty} s_n = -\infty$, A contradiction. So we may assume that $-\infty < \lim_{n\to\infty} t_n < \lim_{n\to\infty} s_n < \infty$. Let

$$A = \lim_{n\to\infty} t_n, B = \lim_{n\to\infty} s_n.$$

Let $\epsilon = \frac{B-A}{4}$. For this $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$A - \epsilon < t_n < A + \epsilon = \frac{B+3A}{4}.$$

Also for this $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$,

$$\frac{A+3B}{4} = B - \epsilon < s_n < B + \epsilon.$$

Since $A < B$,

$$\frac{3A+B}{4} < \frac{B+3A}{4}.$$

Therefore for $n \geq \max\{N_1, N_2\}$, $t_n < s_n$, A contradiciton. □

10. P55. # 9.11

Proof. **(a).** Let

$$B = \inf\{t_n : n \in \mathbb{N}\}.$$

As an infimum of real numbers,

$$B < \infty.$$

So

$$-\infty < B < \infty.$$

Since $\lim_{n\to\infty} s_n = \infty$, for any $M > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$s_n > M - B.$$

Since $B = \inf\{t_n : n \in \mathbb{N}\}$, $t_n \geq B$. Therefore

$$s_n + t_n > M,$$

which implies that $\lim_{n\to\infty}(s_n + t_n) = \infty$.

(b). The case where $\lim_{n\to\infty} t_n < \infty$ is proven similarly as in **(a)**. We discuss the case where $\lim_{n\to\infty} t_n = \infty$. For $M > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$s_n > \frac{M}{2}, t_n > \frac{M}{2}.$$

So we have

$$s_n + t_n > \frac{M}{2} + \frac{M}{2} = M.$$

Therefore we have

$$\lim_{n\to\infty}(s_n + t_n) = +\infty.$$

The claim in **(c)** is proven similarly. □

11. P56. # 9.15

Proof. If $a = 0$, then $\frac{a^n}{n!} = 0$. So $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

If $a \in \mathbb{R}$ and $a \neq 0$, we prove that

$$\lim_{n\to\infty} \left|\frac{a^n}{n!}\right| = \lim_{n\to\infty} \frac{|a|^n}{n!} = 0.$$

We write

$$\frac{|a|^n}{n!} = \frac{|a| \times \cdots \times |a|}{1 \times 2 \times \cdots \times n} = |a| \times \frac{|a|}{2} \times \frac{|a|}{3} \times \cdots \times \frac{|a|}{n}.$$

Then

$$\frac{|a|^n}{n!} \le |a|^{n-1}\frac{|a|}{n}.$$

Therefore

$$\lim_{n\to\infty}\frac{|a|^n}{n!} = 0.$$

□

12. P56. # 9.18

Proof. The proof is skipped. □

13. P64. # 10.1

Proof. In **(a)**, $\frac{1}{n}$ is a decreasing and bounded sequence.

In **(b)**, $\frac{(-1)^n}{n^2}$ is a bounded sequence.

In **(c)**, n^5 is an increasing sequence.

In **(d)**, $\sin\frac{n\pi}{7}$ is a bounded sequence.

In **(e)**, $(-2)^n$ is neither increasing nor decreasing; it is not bounded either.

In **(f)**, $\frac{n}{3^n}$ is a decreasing and bounded sequence. □

14. P65. # 10.3

Proof. We recall that $s_n = K + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n}$ for all n. Since each d_j belongs to the set $\{0, 1, 2, \cdots, 9\}$, we see that $0 \le d_j \le 9$. So

$$s_n \le K + \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = K + 1 - \frac{1}{10^n} < K + 1.$$

□

15. P65. # 10.6

Proof. **(a).** Since $\sum_{n=1}^{\infty} 2^{-n} = 1$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \ge N$,

$$\sum_{n\ge N}\frac{1}{2^n} < \epsilon.$$

For $m > n \geq N$,

(3)

$$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n|$$
$$\leq |s_m - s_{m-1}| + \cdots + |s_{n+1} - s_n|$$
$$\leq \sum_{n \geq N} \frac{1}{2^n} < \epsilon.$$

Therefore we prove that $\{s_n\}$ is a Cauchy sequence. Hence it is convergent.

(b). We assume that $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Then $\{s_n\}$ satisfies the condition

$$|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}.$$

However $\sum_{k=1}^{\infty} \frac{1}{k}$ is a harmonic series.

□

16. P65. # 10.7

Proof. By the completeness Axiom, $\alpha = \sup S$ exists. For any $\frac{1}{n} > 0$, there exists $\alpha - \frac{1}{n}$ is not an upper bound, i.e., there exists $s_n \in S$ such that

$$\alpha - \frac{1}{n} < s_n \leq \alpha \leq \alpha + \frac{1}{n}.$$

By the squeezing theorem,

$$\lim_{n \to \infty} s_n = \alpha = \sup S.$$

□

17. P65. # 10.9

Proof. **(a).** $s_1 = 1, s_2 = \frac{1}{2}, s_3 = \frac{1}{6}, s_4 = \frac{1}{48}$.

(b). Firstly $s_n \geq 0$. Then we prove that $\{s_n\}_{n \geq 1}$ is a decreasing sequence. Then by the monotone convergence theorem, we see that $\lim_{n \to \infty} s_n$ exists.

Indeed,

$$
\begin{aligned}
s_{n+2} - s_{n+1} &= \frac{n+1}{n+2} s_{n+1}^2 - \frac{n}{n+1} s_n^2 \\
&\le \frac{n}{n+1} s_{n+1}^2 - \frac{n+1}{n+2} s_n^2 \\
&= \frac{n+1}{n+2} \left(s_{n+1}^2 - s_n^2\right) \\
&= \frac{n+1}{n+2} (s_{n+1} + s_n)(s_{n+1} - s_n).
\end{aligned} \tag{4}
$$

Since $s_2 - s_1 \le 0$, inductively we prove that $s_{n+1} \le s_n$ for all n. Therefore $\{s_n\}_{n\ge 1}$ is decreasing.

(c). In **(b)**, we have proved that $\alpha = \lim_{n\to\infty} s_n$ exists. Taking limits on both sides of $s_{n+1} = \frac{n}{n+1} s_n^2$, we see that

$$\alpha = \alpha^2,$$

which implies that

$$\alpha = 0, \alpha = 1.$$

Since $s_2 = \frac{1}{2}$ and $\{s_n\}_{n\ge 1}$ is a decreasing sequence, we see

$$\alpha = 0.$$

□

18. P66. # 10.12

Proof. **(a).** $t_n \ge 0$ for all n, and $t_{n+1} \le t_n$ for all $n \in \mathbb{N}$. This proves that $\{t_n\}$ is a bounded and decreasing sequence. Therefore by the monotone convergence theorem, $\lim_{n\to\infty} t_n$ exists.

(b). From **(c)**, $\lim_{n\to\infty} t_n = \frac{1}{2}$.

(c). From $t_{n+1} = \frac{n(n+2)}{(n+1)^2} = \frac{n+2}{n+1} \times \frac{n+1}{n}$, by math induction, we prove that $t_n = \frac{n+1}{2n}$.

Alternatively,

$$
\begin{aligned}
t_n &= \frac{n+1}{n} \times \frac{n-1}{n} \times \frac{n}{n-1} \times \frac{n-2}{n-1} \times \cdots \times t_2 \\
&= \frac{n+1}{n} \times \frac{n-1}{n} \times \frac{n}{n-1} \times \frac{n-2}{n-1} \times \cdots \times \frac{3}{2} \times \frac{1}{2} \\
&= \frac{n+1}{2n}.
\end{aligned} \tag{5}
$$

(d). From **(c)**, $\lim_{n\to\infty} t_n = \frac{1}{2}$.

$\square$

Department of Mathematics, KU, Lawrence, KS 66045

E-mail address: slshao@math.ku.edu