HOMEWORK 7

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ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P239. # 29.1

Proof. **(a).**The mean value theorem holds because x^2 is continuous on $[-1, 2]$ and is differentiable on $(-1, 2)$. Let $f(x) = x^2$. Then

$$f(b) - f(a) = 2^2 - (-1)^2 = 3, b - a = 2 - (-1) = 3.$$

Then since $f'(x) = 2x$, c can be taken as

$$c = \frac{1}{2}.$$

Then

$$f(b) - f(a) = f'(c)(b - a).$$

(c). The mean value theorem does not apply in this case because $|x|$ is not differentiable at 0.

(e). The mean value theorem holds because $\frac{1}{x}$ is continuous on $[1, 3]$ and is differentiable on $(1, 3)$. Let $f(x) = \frac{1}{x}$. Then

$$f(b) - f(a) = \frac{1}{3} - 1 = -\frac{2}{3}, b - a = 3 - 1 = 2.$$

Then since $f'(x) = -\frac{1}{x^2}$, c can be taken as

$$c = \frac{1}{\sqrt{3}}.$$

Then

$$f(b) - f(a) = f'(c)(b - a).$$

□

2. P239. # 29.2

Proof. We know that $\cos' x = -\sin x$ and $|\sin x| \leq 1$ for all $x \in \mathbb{R}$. Then for $x, y \in \mathbb{R}$, there exists $c \in \mathbb{R}$ such that

$$\cos x - \cos y = -\sin c(x-y).$$

Then

$$|\cos x - \cos y| = |\sin c||x-y| \leq |x-y|.$$

This proves the claim. □

3. P239. # 29.3

Proof. **(a).** By the mean value theorem, there exists $x \in (0,2)$ such that

$$f(2) - f(0) = f'(x)(2-0).$$

So

$$f'(x) = \frac{1}{2}.$$

(b). For this one, we will have to invoke Theorem 29.8, the intermediate value theorem for derivatives. So we skip the proof. □

4. P239. #29.4

Proof. We follow the hint. For $h(x) = f(x)e^{g(x)}$, h is continuous on $[a,b]$ and is differentiable on (a,b). Moreover,

$$h(a) = h(b) = 0.$$

By the mean value theorem, there exists $x \in (a,b)$ such that

$$h'(x) = 0.$$

That is to say

$$h'(x) = f'(x)e^{g(x)} + f(x)e^{g(x)}g'(x) = \left(f'(x) + f(x)g'(x)\right)e^{g(x)} = 0.$$

Since $e^{g(x)} \neq 0$ for all x, we obtain

$$f'(x) + f(x)g'(x) = 0.$$

This proves the claim. □

5. P239. # 29.7

Proof. **(a).** Since $f'' = (f')'$ and $f'' \equiv 0$,

$$f' = a$$

for some constant $a \in \mathbb{R}$. We rewrite it,

$$(f(x) - ax)' = 0.$$

Then

$$f(x) - ax = b, \text{i.e., } f(x) = ax + b.$$

for some constant $b \in \mathbb{R}$.

(b). By part (a),

$$f'(x) = ax + b.$$

That is to say,

$$\Big(f(x) - \frac{a}{2}x^2 - bx\Big)' = 0.$$

Then

$$f(x) - \frac{a}{2}x^2 - bx = c$$

for some constant c. Then

$$f(x) = \frac{a}{2}x^2 + bx + c.$$

□

6. P240. # 29.9

Proof. This is obviously true for $x \leq 0$. We consider $f(x) = e^x - x$ on $[0, \infty)$ Then

$$f'(x) = e^x - 1 \geq 0.$$

This holds for $x \geq 0$. So f is increasing on $[0, \infty)$:

$$f(x) \geq f(0) = 1 \geq 0.$$

Then $e^x \geq x$. □

7. P240. # 29.11

Proof. Consider $f(x) = x - \sin x$. Then $f'(x) = 1 - \cos x \geq 0$ for all x. So f is increasing on $[0, \infty)$.

$$f(x) \geq f(0) = 0.$$

So

$$x \geq \sin x.$$

□

8. P240. # 29.18

Proof. **(a).**By mean value theorem,

$$s_{n+1} - s_n = f(s_n) - f(s_{n-1}) = f'(c)(s_n - s_{n-1}).$$

Then

$$|s_{n+1} - s_n| = |f'(c)(s_n - s_{n-1})| \leq a|s_n - s_{n-1}|.$$

Therefore,

$$|s_{n+1} - s_n| \leq a^n|s_1 - s_0|.$$

So for $m > n$,

$$\begin{aligned}|s_m - s_n| &\leq |s_m - s_{m-1}| + a|s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| \\ &\leq (a^{m-1} + a^{m-2} + \cdots + a^n)|s_1 - s_0| \\ &\leq \frac{a^n - a^m}{1-a}|s_1 - s_0| \\ &= a^n \frac{1}{1-a}|s_1 - s_0|.\end{aligned}$$

We assume that $|s_1 - s_0| \neq 0$. Since $\lim_{n\to\infty} a^n = 0$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$|a^n| \leq \frac{(1-a)\epsilon}{|s_1 - s_0|}.$$

So

$$|s_m - s_n| < \epsilon.$$

This proves that $\{s_n\}$ is Cauchy. Hence s_n is a convergent sequence. This proves (a).

(b). Let $s = \lim_{n\to\infty} s_n$. We take $n \to \infty$ in $s_n = f(s_{n-1})$. Then because f is differentiable and so continuous on $\mathbb{R}$, we obtain

$$s = f(s).$$

□

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