HOMEWORK 8

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ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P279. # 32.1

Proof. Given a partition of $[a, b]$: $a + t_0 < t_1 < \cdots < t_n$. Since $f(x) = x^3$ is increasing on $[t_{k_1}, t_k]$. Then

$$M(f, [t_{k-1}, t_k]) = t_k^3, m(f, [t_{k-1}, t_k]) = t_{k-1}.$$

Then the upper and lower Darboux sums,

$$U(f, p) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} t_k^3 (t_k - t_{k-1}),$$

and

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} t_{k-1}^3 (t_k - t_{k-1}).$$

We take $t_k = \frac{kb}{n}$ for $1 \le k \le n$. Then

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} \frac{k^3 b^3}{n^3} \frac{b}{n} = \frac{b^4}{n^4} \sum_{k=1}^{n} k^3 \\ &= \frac{b^4}{n^4} (1 + 2 + \cdots + n)^2 = \frac{b^4}{n^4} \frac{n^2 (n+1)^2}{4} \\ &= \frac{(n+1)^2 b^4}{4n^2}. \end{aligned}$$

By the definition of upper integrals,

$$U(f) \le \frac{(n+1)^2 b^4}{4n^2}$$

Let $n \to \infty$,

$$U(f) \le \frac{b^4}{4}.$$

Similarly we compute that

$$L(f) \ge \frac{(n-1)^2 b^4}{4n^2}.$$

Let $n \to \infty$,

$$L(f) \geq \frac{b^4}{4}.$$

Since $L(f) \leq U(f)$, we obtain

$$L(f) = U(f) = \frac{b^4}{4}.$$

$\square$

2. P279. # 32.2

Proof. **(a).**By the same reasoning as in Example 2, for any partition $P = \{t_0 = 0 < t_1 < \cdots < t_n = b\}$,

$$L(f, P) = 0, \text{ and } L(f) = 0.$$

We compute the upper Darboux sum,

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}).$$

For $\{t_k\}$ rational numbers, $M(f, [t_{k-1}, t_k]) = t_k$ and so

$$U(f, P) = \sum_{k=1}^{n} t_k(t_k - t_{k-1}).$$

For $\{t_k\}$ irrational numbers, $M(f, [t_{k-1}, t_k]) = t_k$ and so

$$U(f, P) = \sum_{k=1}^{n} t_k(t_k - t_{k-1}).$$

We take $t_k = \frac{kb}{n}$. Then

$$U(f, P) = \sum_{k=1}^{n} \frac{kb^2}{n^2} = \frac{n(n+1)}{2n^2} b^2 = \frac{n+1}{2n} b^2.$$

So

$$U(f) = \inf\{U(f, P) : P \text{ is a partition}\} \leq \frac{n+1}{2n} b^2, \text{ for all } n.$$

Then

$$U(f) \leq \frac{b^2}{2}.$$

(b). $\square$

3. P279. # 32.3

Proof. The proof is similar as in Exercise 32.2. $\square$

4. P279. # 32.4

Proof. Suppose $Q \subset P$ and $Q \neq P$. We further assume that $Q = P \cup \{y_1, \cdots, y_m\}$. We do the induction on m. When $m = 1$, this is the argument in the proof of Lemma 32.2. Suppose that $m = k$, the conclusion holds. When $m = k+1$, Q contains one more point than $P \cup \{y_1, \cdots, y_k\}$ for some $y_1, \cdots, y_k in \mathbb{R}$. This argument is similar to that when Q contains one more point than P. $\square$

5. P279. # 32.6

Proof. By the definition of the upper and lower integrals,

$$U(f) \leq U_n, L(f) \geq L_n$$

for any n. Then

$$0 \leq U(f) - L(f) \leq U_n - L_n.$$

By the squeezing theorem,

$$U(f) = L(f),$$

So f is integrable.

Since $L_n \leq \int_a^b f \leq U_n$, then

$$0 \leq U_n - \int_a^b f \leq U_n - L_n.$$

By the squeezing theorem again,

$$\lim_{n\to\infty} U_n - \int_a^b f = 0.$$

That is to say,

$$\lim_{n\to\infty} U_n = \int_a^b f.$$

Similarly,

$$\lim_{n\to\infty} L_n = \int_a^b f.$$

$\square$

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