Math 800 Final Exam
Spring 2017

Note: These 10 problems are taken from the complex analysis part of UCLA qualifying exams in Analysis for fall 2016 and spring 2016 at *https://secure.math.ucla.edu/gradquals/hbquals.php.* Due by Wednesday, May 3rd in class.

1) Determine

$$\int_0^\infty \frac{x^{a-1}}{x+z}dx$$

for $0 < a < 1$ and $\operatorname{Re} z > 0$. Justify all manipulations.

2) Let $\mathbf{C}_+ = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ and let $f_n : \mathbf{C}_+ \to \mathbf{C}_+$ be a sequence of holomorphic functions. Show that unless $|f_n| \to \infty$ uniformly on compact subsets of $\mathbf{C}_+$, there exists a subsequence converging uniformly on compact subsets of $\mathbf{C}_+$.

3)Let $f : \mathbf{C} \to \mathbf{C}$ be entire and assume that $|f(z)| = 1$ when $|z| = 1$. Show that f is in the following form,

$$f(z) = Cz^m$$

for some integer $m > 0$ and $C \in \mathbf{C}$ with $|C| = 1$.

4) Does there exist a function $f(z)$ holomorphic in the disk $|z| < 1$ such that $\lim_{|z|\to 1} |f(z)| = \infty$? Either find one or prove that none exist.

5) Assume that $f(z)$ is holomorphic on $|z| < 2$. Show that

$$\max_{|z|=1} \left| f(z) - \frac{1}{z} \right| \geq 1.$$

6)

(a). Find a real-valued harmonic function v defined on the disk $|z| < 1$ such that $v(z) > 0$ and $\lim_{z\to 1} v(z) = \infty$.

(b). Let u be a real-valued harmonic function in the disk $|z| < 1$ such that $u(z) \leq M < \infty$ and $\limsup_{r\to 1} u(re^{i\theta}) \leq 0$ for all $\theta \in (0, 2\pi)$. Show that $u(z) \leq 0$. The function in part (a) is useful here.

7) Let $\mathcal{H}$ be the space of holomorphic functions f in $D = \{z \in \mathbf{C} : |z| < 1\}$ such that

$$\int_D |f(z)|^2 dA(z) < \infty,$$

where the integration is with respect to the Lebesgue measure A on D. The vector space $\mathcal{H}$ is a Hilbert space if equipped with the inner product

$$\langle f, g\rangle = \int_D f(z)\overline{g}(z)dA(z)$$

for $f, g \in \mathcal{H}$. Fix $z_0 \in D$ and define $L_{z_0}(f) = f(z_0)$ for $f \in \mathcal{H}$.

(a). Show that L_{z_0} is a bounded linear functional on $\mathcal{H}$.

(b). Find an explicit $g_{z_0} \in \mathcal{H}$ such that

$$L_{z_0}(f) = \langle f, g_{z_0} \rangle$$

for all $f \in \mathcal{H}$.

8) Let f be a continuous complex-valued function on the closed unit disk $\bar{D} = \{z \in \mathbf{C} : |z| \leq 1\}$ such that f is holomorphic in the open disk $D = \{z \in \mathbf{C} : |z| < 1\}$ and $f(0) \neq 0$. Show that

(a). Let $0 < r < 1$ and $\inf_{|z|=r} |f(z)| > 0$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \geq \log |f(0)|.$$

(b). The measure $|\{\theta \in [0, 2\pi] : f(e^{i\theta}) = 0\}| = 0$, where $|E|$ denotes the Lebesuge measure of $E \subset [0, 2\pi]$.

9) Let μ be a positive Borel measure on $[0, 1]$ with $\mu([0, 1]) = 1$.

(a). Show that the function $f(z)$ defined as

$$f(z) = \int_{[0,1]} e^{izt} d\mu(t)$$

for $z \in \mathbf{C}$ is holomorphic on $\mathbf{C}$.

(b). Suppose that there exists $n \in \mathbb{N}$ such that

$$\limsup_{|z| \to \infty} \frac{|f(z)|}{|z|^n} < \infty.$$

Show that μ equals the Dirac measure δ_0 at 0.

9) Show that $f : \mathbf{C} \to \mathbf{C}$ is holomorphic function such that the function

$$z \mapsto g(z) = f(z) f(1/z)$$

is bounded on $\mathbf{C} \setminus \{0\}$.

(a). Show that if $f(0) \neq 0$, then f is a constant.

(b). Show that if $f(0) = 0$, then there exists $a \in \mathbf{C}$ and $n \in \mathbf{N}$ such that $f(z) = az^n$ for all $z \in \mathbf{C}$.