Math 500, Final

Name (Print): (first)____________ (last)____________

Signature:

There are a total of 100+20 points on this 2 hours and 30 minutes' exam. This contains 12 pages (including this cover page) and 10+ 1 problems. Check to see if any page is missing. Enter all requested information on the top of this page. Please turn off cell phones. You are allowed to bring one single-sided 8.5×11 inch page of notes (the print paper size), in your own handwriting, to the exam. Do not give numerical approximations to quantities such as $\sin 5, \pi, e$ or $\sqrt{2}$. However you should simplify $\sin \frac{\pi}{2} = 1$ and $e^0 = 1$, etc.

The following rules apply:

- To get full credit for a problem you must show the details of your work, in a reasonably neat and coherent way, in the space provided. Answers unsupported by an argument will get little credit. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- Mysterious or unsupported answers will not receive full credit. Your work should be mathematically CORRECT and carefully and legibly written.
- NO books. No computers. No calculators. Do all of your calculations on this test paper.

Problem	1	2	3	4	5	6	7	8	9	10	11
Score											

Problem 1 (a). (7 points). Suppose $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Use the $\varepsilon - N$ formulation of limits to prove that

$$\lim_{n\to\infty} (a_n + b_n) = A + B.$$

Problem 1 (b). (3 points). Use the theorem in Part (a) to evaluate

$$\lim_{n\to\infty} \left(1 - \frac{|\sin n|}{n}\right).$$

Problem 2. The monotone convergence theorem says that, every monotone increasing sequence which is bounded above is convergent. Use this theorem to evaluate the limit of the sequence

$$a_1 = 0, a_{n+1} = \frac{a_n}{2} + 1, \text{ for } n \in \mathcal{N}$$

by following the steps:

(a). (4 points). Show that $\{a_n\}$ is monotone increasing.

(b). (4 points). Show that $a_n \leq 2$ for all $n \in \mathcal{N}$.

(c). (2 points). Evaluate $A = \lim_{n\to\infty} a_n$.

Problem 3. (10 points). Determine whether the following series converge or not? Justify your answers.

a). $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$.

b). $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

c). $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

d). $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$.

e). $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)^2$.

Problem 4. (10 points). For which values of x does the power series $\sum_{n=1}^{\infty} c_n x^n$ converge?

a). $c_n = \frac{1}{\sqrt{n}}, n = 1, 2, \cdots$

b). $c_n = \frac{1}{n^2}, n = 1, 2, \cdots$

c). $c_n = \frac{1}{n^n}, n = 1, 2, \cdots$

Problem 5 a). (5 points.) Evaluate

$$\lim_{x \to -5} \frac{x+5}{x^2 - 25}.$$

5 b). (5 points.) Show that $\lim_{x \to 0} \sin \dfrac{1}{x}$ does not exist.

Problem 6). (10 points.) Prove that the equation $x^4 - 5x^3 + x - 1 = 0$ has a solution in $[-1, 0]$.

Problem 7 a). (5 points.) Prove that $f(x) = |x^3|$ is differentiable at 0.

7 b). (5 points.) Prove that $g(x) = |x|$ is not differentiable at 0.

Problem 8). Prove Rolle's theorem: suppose that f is differentiable on $[a, b]$ and $f(a) = f(b)$, and f is not a constant function. Following the following steps to prove that there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

a). (3 points.) Use the maximum principle to show that either there exists $c \in (a, b)$ such that $f(c) = \max_{x\in[a,b]} f(x)$, or $d \in (a, b)$ such that $f(d) = \min_{x\in[a,b]} f(x)$.

b). (7 points.) Suppose that the first alternative occurs in **Part a)**, show that $f'(c) = 0$.

Problem 9 a). (5 points.) Prove that $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

9 b). (5 points.) Use the $\varepsilon - \delta$ formulation of uniform continuity to prove that $f(x) = \sqrt{x}$ is uniformly continuous on $(0, \infty)$.

Problem 10 a). (5 points.) Let

$$f(x) = \begin{cases} 1, \quad 0 \le x \le 1, \\ x+1, 1 < x < 2, \\ e^x, 2 \le x \le 4. \end{cases}$$

Calculate the integral $\int_0^4 f(x)dx$.

Problem 10 b). (5 points.) Let

$$f(x) = \begin{cases} 1, \text{ for irrational } x, \\ 0, \text{ for rational } x. \end{cases}$$

Prove that f is not Riemann integrable on $[0, 1]$ (Hint: show $L(f) \neq U(f)$).

Problem 11). (10 points.) Let $f : [a, b] \to \mathbf{R}$ be Riemann integrable and continuous at a point $c \in (a, b)$. Define

$$F(x) = \int_a^x f(t)dt, \text{ for all } x \in [a, b].$$

Prove that

$$F'(c) = f(c).$$