Math 500, Final

Name (Print): (first)____________ (last)____________

Signature:

There are a total of 100+20 points on this 2 hours and 30 minutes' exam. This contains 12 pages (including this cover page) and 10+ 1 problems. Check to see if any page is missing. Enter all requested information on the top of this page. Please turn off cell phones. You are allowed to bring one single-sided 8.5×11 inch page of notes (the print paper size), in your own handwriting, to the exam. Do not give numerical approximations to quantities such as $\sin 5, \pi, e$ or $\sqrt{2}$. However you should simplify $\sin \frac{\pi}{2} = 1$ and $e^0 = 1$, etc.

The following rules apply:

- To get full credit for a problem you must show the details of your work, in a reasonably neat and coherent way, in the space provided. Answers unsupported by an argument will get little credit. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- Mysterious or unsupported answers will not receive full credit. Your work should be mathematically CORRECT and carefully and legibly written.
- NO books. No computers. No calculators. Do all of your calculations on this test paper.

Problem	1	2	3	4	5	6	7	8	9	10	11
Score											

Problem 1 (a). (7 points). Suppose $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Use the $\varepsilon - N$ formulation of limits to prove that

$$\lim_{n\to\infty}(a_n + b_n) = A + B.$$

Proof. Since $\lim_{n\to\infty} a_n = A$, for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N$,

$$|a_n - A| < \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} b_n = B$, for the same $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - B| < \frac{\varepsilon}{2}.$$

Hence we take $N = \max\{N_1, N_2\}$ for any $n \geq N$,

$$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that

$$\lim_{n\to\infty} a_n + b_n = A + B.$$

□

Problem 1 (b). (3 points). Use the theorem in Part (a) to evaluate

$$\lim_{n\to\infty}\left(1 - \frac{|\sin n|}{n}\right).$$

Proof. Since $0 \leq \frac{|\sin n|}{n} \leq \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, by the squeezing theorem for limits, we have

$$\lim_{n\to\infty} \frac{|\sin n|}{n} = 0.$$

By the claim proved above,

$$\lim_{n\to\infty}\left(1 - \frac{|\sin n|}{n}\right) = 1 - 0 = 1.$$

□

Problem 2. The monotone convergence theorem says that, every monotone increasing sequence which is bounded above is convergent. Use this theorem to evaluate the limit of the sequence

$$a_1 = 0, a_{n+1} = \frac{a_n}{2} + 1, \text{ for } n \in \mathcal{N}$$

by following the steps:

(a). (4 points). Show that $\{a_n\}$ is monotone increasing.

Proof. We prove it by math induction. It is known that $a_1 = 1$, $a_2 = \frac{3}{2}$. So

$$a_1 \leq a_2.$$

We assume that $a_n \leq a_{n+1}$. Then

$$a_{n+1} = \frac{a_n}{2} + 1, a_{n+2} = \frac{a_{n+1}}{2} + 1.$$

Hence

$$a_{n+1} \leq a_{n+2}.$$

This proves that $\{a_n\}$ is monotone increasing. □

(b). (4 points). Show that $a_n \leq 2$ for all $n \in \mathcal{N}$.

Proof. We prove it by math induction. It is known that $a_1 = 1 \leq 2$. We assume that $a_n \leq 2$. Since $a_{n+1} = \frac{a_n}{2} + 1$, we have

$$a_{n+1} \leq \frac{a_n}{2} + 1 \leq 2.$$

This proves that $a_n \leq 2$ for all n. □

(c). (2 points). Evaluate $A = \lim_{n\to\infty} a_n$.

Proof. Since $\{a_n\}$ is monotone increasing and bounded by (a) and (b), by monotone convergence theorem,

$$A = \lim_{n\to\infty} a_n \text{ exists.}$$

Taking limits on both sides of $a_{n+1} = \frac{a_n}{2} + 1$, we get

$$A = \frac{A}{2} + 1.$$

So $A = 2$. □

Problem 3. (10 points). Determine whether the following series converge or not? Justify your answers.

a). $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$.

Proof. It converges because both $\sum \frac{1}{n^3}$ and $\sum \frac{1}{3^n}$ converge. □

b). $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$.

Proof. It diverges. We compare it with the integral $\int_2^{\infty} \frac{1}{x\ln x}dx$.

$$\int_2^{\infty} \frac{1}{x\ln x}dx = \int_{\ln 2}^{\infty} \frac{1}{t}dt = \infty.$$

□

c). $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

Proof. It converges. We compare it with the integral $\int_2^{\infty} \frac{1}{x(\ln x)^2}dx$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}dx = \int_{\ln 2}^{\infty} \frac{1}{t^2}dt = -\frac{1}{t}|_{\ln 2}^{\infty} = \frac{1}{\ln 2}.$$

□

d). $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$.

Proof. It converges by the theorem of the alternating series. □

e). $\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^2$.

Proof. It diverges.

$$\lim_{n\to\infty}(1+\frac{1}{n^2})^2 = 1.$$

□

Problem 4. (10 points). For which values of x does the power series $\sum_{n=1}^{\infty} c_n x^n$ converge?

a). $c_n = \frac{1}{\sqrt{n}}, n = 1, 2, \cdots$

Proof. By the ratio test,

$$q = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = 1.$$

Hence the radius of convergence is

$$\rho = \frac{1}{q} = 1.$$

Hence for $|x| < 1$, the series $\sum c_n x^n$ converges. For $x = -1$, by the theorem of alternating series, it converges too. For $x = 1$, the series reduces to $\sum \frac{1}{\sqrt{n}}$, which diverges.

Therefore for $x \in [-1, 1)$, the series converges. □

b). $c_n = \frac{1}{n^2}, n = 1, 2, \cdots$

Proof. By the ratio test,

$$q = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{n^2}{(n+1)^2} = 1.$$

Hence the radius of convergence is

$$\rho = \frac{1}{q} = 1.$$

Hence for $|x| < 1$, the series $\sum c_n x^n$ converges. For $x = \pm 1$,

$$|c_n x^n| = \frac{1}{n^2},$$

the series $\sum c_n x^n$ converges. Therefore for $x \in [-1, 1]$, the series converges. □

c). $c_n = \frac{1}{n^n}, n = 1, 2, \cdots$

Proof. We know that

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)^{n+1}}{n^n} = \frac{(n+1)^n}{n^n} \times (n+1) \geq n + 1.$$

So $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \infty$. Therefore the series $\sum c_n x^n$ only converges at $x = 0$. □

Problem 5 a). (5 points.) Evaluate

$$\lim_{x \to -5} \frac{x+5}{x^2-25}.$$

Proof.

$$\lim_{x \to -5} \frac{x+5}{x^2-25} = \lim_{x \to -5} \frac{1}{x-5} = -\frac{1}{10}.$$

□

5 b). (5 points.) Show that $\lim_{x \to 0} \sin \dfrac{1}{x}$ does not exist.

Proof. Taking $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Then

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0.$$

However

$$\lim_{n \to \infty} \sin \frac{1}{x_n} = \lim_{n \to \infty} \sin 2n\pi = 0;$$

and

$$\lim_{n \to \infty} \sin \frac{1}{y_n} = \lim_{n \to \infty} \sin(2n\pi + \frac{\pi}{2}) = 1.$$

So the limit does not exist. □

Problem 6). (10 points.) Prove that the equation $x^4 - 5x^3 + x - 1 = 0$ has a solution in $[-1, 0]$.

Proof. Let $f(x) = x^4 - 5x^3 + x - 1$. We know that

$$f(-1) = (-1)^4 - 5(-1)^3 + x - 1 = 4 \text{ and } f(0) = -1.$$

So by the intermediate value theorem, there exists $c \in [-1, 0]$ such that

$$f(c) = 0.$$

That is to say, $c^4 - 5c^3 + c - 1 = 0$. □

Problem 7 a). (5 points.) Prove that $f(x) = |x^3|$ is differentiable at 0.

Proof. For $h \neq 0$,

$$\frac{f(0+h) - f(0)}{h} = \frac{|h^3|}{h} = \frac{h^2|h|}{h} = h|h|.$$

Hence

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

This proves that f is differentiable at 0 and $f'(0) = 0$. □

7 b). (5 points.) Prove that $g(x) = |x|$ is not differentiable at 0.

Proof.

$$\frac{g(0+h) - g(0)}{h} = \frac{|h|}{h}.$$

Choose $h_n > 0$,

$$\lim_{n \to \infty} \frac{g(0+h_n) - g(0)}{h_n} = 1,$$

while if choosing $h_n < 0$,

$$\lim_{n \to \infty} \frac{g(0+h_n) - g(0)}{h_n} = -1.$$

So $\lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$ does not exist. □

Problem 8). Prove Rolle's theorem: suppose that f is differentiable on $[a, b]$ and $f(a) = f(b)$, and f is not a constant function. Following the following steps to prove that there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

a). (3 points.) Use the maximum principle to show that either there exists $c \in (a, b)$ such that $f(c) = \max_{x\in[a,b]} f(x)$, or $d \in (a, b)$ such that $f(d) = \min_{x\in[a,b]} f(x)$.

Proof. Since f is differentiable on $[a, b]$, f is continuous on $[a, b]$. By the maximum principle, there exists $c, d \in [a, b]$ such that

$$f(c) = \sup_{x\in[a,b]} f(x), \quad f(d) = \inf_{x\in[a,b]} f(x).$$

If $f(c) = f(d)$, the function f is constant. By the condition, $f(c) > f(d)$. Since $f(a) = f(b)$, the points c, d are not endpoints a, b at the same time. So either $c \in (a, b)$ or $d \in (a, b)$. □

b). (7 points.) Suppose that the first alternative occurs in **Part a)**, show that $f'(c) = 0$.

Proof. If $f(c) = \sup_{x\in[a,b]} f(x)$ and $c \in (a, b)$, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$.

For $0 < h < \delta$, $f(c) \geq f(c + h)$,

$$\frac{f(c + h) - f(c)}{h} \leq 0.$$

Since f is differentiable at c,

$$f'(c) = \lim_{h\to 0} \frac{f(c + h) - f(c)}{h} \leq 0. \tag{1}$$

On the other hand, for $-\delta < h < 0$, $f(c) \geq f(c + h)$, and

$$\frac{f(c + h) - f(c)}{h} \geq 0.$$

Since f is differentiable at c,

$$f'(c) = \lim_{h\to 0} \frac{f(c + h) - f(c)}{h} \geq 0. \tag{2}$$

By (1) and (2),

$$f'(c) = 0.$$

□

Problem 9 a). (5 points.) Prove that $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

Proof. For $a \in (0, \infty)$,

$$\sqrt{x} - \sqrt{a} = \frac{x - a}{\sqrt{x} + \sqrt{a}} \leq \frac{x - a}{\sqrt{a}}.$$

So for any $\varepsilon > 0$, choosing $\delta = \varepsilon\sqrt{a}$, then for $|x - a| < \delta$,

$$|\sqrt{x} - \sqrt{a}| < \frac{\delta}{\sqrt{a}} = \varepsilon.$$

This proves that f is continuous at a. □

9 b). (5 points.) Use the $\varepsilon - \delta$ formulation of uniform continuity to prove that $f(x) = \sqrt{x}$ is uniformly continuous on $(0, \infty)$.

Proof. We give two proofs for this exercise.

I. We split $(0, \infty) = (0, 2) \cup [1, \infty)$. On the interval $[1, \infty)$, by the mean value theorem, there exists $c \in (x, y)$ or (y, x) such that

$$f(x) - f(y) = f'(c)(x - y) = \frac{1}{2\sqrt{c}}(x - y).$$

Since $c \in [1, \infty)$, $\sqrt{c} \geq 1$. Hence

$$|f(x) - f(y)| \leq \frac{|x - y|}{2}.$$

For $\varepsilon > 0$, there exists $\delta_1 = 2\varepsilon$ such that for $|x - y| < \delta$,

$$|f(x) - f(y)| < \frac{\delta_1}{2} = \varepsilon.$$

This proves that when restricted to the interval $[1, \infty)$, the function f is uniformly continuous.

On the interval $[0, 2]$, the function $\sqrt{x}$ is continuous and hence uniformly continuous. That is to say, for the same $\varepsilon > 0$, there exists $\delta_2 > 0$ such that if $|x - y| < \delta_2$,

$$|f(x) - f(y)| < \varepsilon.$$

So when restricted to the interval $(0, 2]$, f is uniformly continuous.

Now we conclude that, for $\varepsilon > 0$, there exists $\delta = \min\{\delta_1, \delta_2, \frac{1}{2}\}$, if $|x - y| < \delta$, i.e., either $x, y \in (0, 2]$ or $x, y \in [1, \infty)$,

$$|f(x) - f(y)| < \varepsilon.$$

This proves that f is uniformly continuous on $(0, \infty)$.

II. We observe that for $x, y \in (0, \infty)$, if $x \geq y$,

$$\sqrt{x} - \sqrt{y} \leq \sqrt{x - y}.$$

This is proved by squaring both sides of $\sqrt{x} \leq \sqrt{x-y} + \sqrt{y}$. On the other hand, if $x < y$,

$$\sqrt{y} - \sqrt{x} \leq \sqrt{|y - x|}.$$

Thus in any case, for $x, y \in (0, \infty)$,

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{x - y}.$$

For any $\varepsilon > 0$, there exists $\delta = \varepsilon^2$, for any $|x - y| < \delta$,

$$|\sqrt{x} - \sqrt{y}| < \sqrt{\delta} = \varepsilon.$$

Thus f is uniformly continuous on $(0, \infty)$. □

Problem 10 a). (5 points.) Let

$$f(x) = \begin{cases} 1, \quad 0 \leq x \leq 1, \\ x + 1, 1 < x < 2, \\ e^x, 2 \leq x \leq 4. \end{cases}$$

Calculate the integral $\int_0^4 f(x)dx$.

Proof.

$$\int_0^4 f(x)dx = \int_0^1 1dx + \int_1^2 (x+1)dx + \int_2^4 e^x dx = e^4 - e^2 + \frac{7}{2}.$$

□

Problem 10 b). (5 points.) Let

$$f(x) = \begin{cases} 1, & \text{for irrational } x, \\ 0, & \text{for rational } x. \end{cases}$$

Prove that f is not Riemann integrable on $[0,1]$ (Hint: show $L(f) \neq U(f)$).

Proof. For any partition $P = \{x_0, x_1, \cdots, x_n\}$,

$$U_P(f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} 1 \times (x_k - x_{k-1}) = 1;$$

and

$$L_P(f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} 0 \times (x_k - x_{k-1}) = 0;$$

This is because, on any subinterval $[c,d]$ of $[0,1]$, it always contains an irrational number and a rational number. So

$$\sup_{x \in [c,d]} f(x) = 1, \quad \inf_{x \in [c,d]} f(x) = 0.$$

Furthermore,

$$L(f) = 0, \quad U(f) = 1.$$

This proves that $L(f) \neq U(f)$. Hence f is not integrable. $\square$

Problem 11). (10 points.) Let $f : [a, b] \to \mathbf{R}$ be Riemann integrable and continuous at a point $c \in (a, b)$. Define

$$F(x) = \int_a^x f(t)dt, \text{ for all } x \in [a, b].$$

Prove that

$$F'(c) = f(c).$$

Proof. For $c \in (a, b)$, there exists $\delta_1 > 0$ such that $(c - \delta_1, c + \delta_1) \subset (a, b)$.

For $h \in (-\delta_1, \delta_1)$ and $h \neq 0$, we consider

$$\frac{F(c+h) - F(c)}{h} = \frac{1}{h}\int_c^{c+h} f(x)dx.$$

Since f is continuous at c, for any $\varepsilon > 0$, there exists $\delta_2 > 0$ such that if $|x - c| < \delta_2$,

$$|f(x) - f(c)| \leq \varepsilon.$$

So taking $\delta = \min\{\delta_1, \delta_2\}$. For $h \in (0, \delta)$,

$$\frac{1}{h}\int_c^{c+h} f(x)dx - f(c) \;=\; \frac{1}{h}\int_c^{c+h}(f(x) - f(c))dx \;\leq \frac{1}{h}\int_c^{c+h}|f(x) - f(c)|dx < \varepsilon.$$

This applies to any positive sequence $\{h_n\}$ in $(0, \delta)$. So

$$\lim_{h\to 0}\frac{1}{h}\int_c^{c+h} f(x)dx - f(c) = 0. \tag{3}$$

On the other hand, if $h \in (-\delta, 0)$,

$$\frac{1}{h}\int_c^{c+h} f(x)dx - f(c) \;=\; \frac{1}{h}\int_c^{c+h}(f(x) - f(c))dx \;\leq \frac{1}{-h}\int_{c+h}^{c}|f(x) - f(c)|dx < \varepsilon.$$

This applies to any negative sequence $\{h_n\}$ in $(-\delta, 0)$. So

$$\lim_{h\to 0}\frac{1}{h}\int_c^{c+h} f(x)dx - f(c) = 0. \tag{4}$$

By (3) and (4),

$$\lim_{h\to 0}\frac{1}{h}\int_c^{c+h} f(x)dx = f(c).$$

This proves that F is differentiable at c, and $F'(c) = f(c)$. □