

HOMWORK 2

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ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P38. #7.3

Proof. The sequences in **(b)**, **(d)**, **(f)**, **(j)**, **(p)**, **(r)**, **(t)** converge; their limits are $1, 1, 1, \frac{7}{2}, 2, 1$ and 0 , respectively.

The limit in **(h)** diverges because it is not bounded.

The limit in **(l)** diverges because it is a sequence consisting of $0, 1, -1$; the limit in **(n)** diverges because it is a sequence consisting of $0, \frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{2}$. You can refer to Example 4 in Section 8 for a discussion. \square

P38. # 7.4

Proof. **(a).** $x_n = \frac{1}{n\sqrt{2}}$; Then

$$\lim_{n \rightarrow \infty} x_n = 0.$$

(b). $r_n = \sum_{k=1}^n \frac{1}{k^2}$. Then

$$\lim_{n \rightarrow \infty} r_n = \frac{\pi^2}{6}.$$

\square

2. P44. # 8.2

Proof. **(b).** $\lim_{n \rightarrow \infty} b_n = \frac{7}{3}$. For any $\epsilon > 0$, we need to find $N \in \mathbb{N}$ such that for any $n \geq N$,

$$(1) \quad \frac{7n - 19}{3n + 7} - \frac{7}{3} < \epsilon.$$

In order for (3) to hold,

$$n > \frac{106}{9\epsilon} - \frac{7}{3}.$$

So we take $N \in \mathbb{N}$ and $N > \frac{106}{9\epsilon} - \frac{7}{3}$. To conclude, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\frac{7n - 19}{3n + 7} - \frac{7}{3} < \epsilon.$$

Similarly in (d),

$$\lim_{n \rightarrow \infty} d_n = \frac{2}{5}.$$

□

3. P44. # 8.4

Proof. $\lim_{n \rightarrow \infty} s_n = 0$: for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$|s_n - 0| \leq \frac{\epsilon}{M + 1}.$$

Then

$$|s_n t_n - 0| \leq |t_n| |s_n| \leq M \frac{\epsilon}{M + 1} < \epsilon.$$

This proves that $\lim_{n \rightarrow \infty} s_n t_n = 0$.

□

4. P44. # 8.5(A)(B).

Proof. (a). This is proven in class. Please refer to the class notes.

(b). $\lim_{n \rightarrow \infty} t_n = 0$: for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|t_n - 0| = t_n < \epsilon.$$

Since $|s_n| \leq t_n$,

$$|s_n - 0| < \epsilon.$$

Therefore

$$\lim_{n \rightarrow \infty} s_n = 0.$$

□

5. P44. # 8.6

Proof. (a) (b). This is proven in class. Please refer to the class notes. □

6. P45. # 8.8

Proof. (a).

$$0 \leq \sqrt{n^2 + 1} - n = \frac{1}{\sqrt{n^2 + 1} + n} \leq \frac{1}{n}.$$

Since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$,

$$\lim_{n \rightarrow \infty} \sqrt{n^2 + 1} - n = 0.$$

(b). For any $\epsilon > 0$, we need to find $N \in \mathbb{N}$ such that for any $n \geq N$,

$$(2) \quad |(\sqrt{n^2 + n} - n) - \frac{1}{2}| < \epsilon.$$

We know that

$$|(\sqrt{n^2 + n} - n) - \frac{1}{2}| = \frac{n}{\sqrt{n^2 + n} + n} - \frac{1}{2} = \frac{n}{2(\sqrt{n^2 + n} + n)^2} \leq \frac{1}{2n}.$$

In order for (2) to hold, it suffices that

$$\frac{1}{2n} < \epsilon, \text{ i.e., } n > \frac{1}{2\epsilon}.$$

We can take $N \in \mathbb{N}$ and $N > \frac{1}{2\epsilon}$.

The proof in (c) is similar. □

7. P54. # 9.4

Proof. (a). $s_1 = 1, s_2 = \sqrt{2}, s_3 = \sqrt{\sqrt{2} + 1}, s_4 = \sqrt{\sqrt{\sqrt{2} + 1}}$.

(b). Let $A = \lim_{n \rightarrow \infty} s_n$. Then taking limits on both sides of $s_{n+1} = \sqrt{s_n + 1}$.

$$A = \sqrt{A + 1}.$$

Therefore

$$A^2 - A - 1 = 0.$$

So $A = \frac{1 \pm \sqrt{5}}{2}$. Since $A \geq 0$, we see that

$$A = \frac{1 + \sqrt{5}}{2}.$$

□

8. P54. # 9.5

Proof. Let $A = \lim_{n \rightarrow \infty} t_n$. Obviously by math induction $t_n \geq 0$; then $t_{n+1} \geq \frac{2\sqrt{2}t_n}{2t_n} \geq \sqrt{2}$. Taking limits on both sides of $t_{n+1} = \frac{t_n^2+2}{2t_n}$, we see that

$$A = \frac{A^2 + 2}{2A}.$$

This proves that

$$A = \sqrt{2}.$$

□

9. P55. #9.9

Proof. (a). If $\lim_{n \rightarrow \infty} s_n = \infty$, for any $M > 0$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$,

$$s_n \geq M.$$

Taking $n \geq \max\{N_0, N_1\}$ and then we see that

$$t_n \geq M.$$

This proves that $\lim_{n \rightarrow \infty} t_n = \infty$.

The claim in **(b)** is proven similarly.

(c). For the claim in **(c)**, we prove it by contradiction. Suppose that $\lim_{n \rightarrow \infty} s_n > \lim_{n \rightarrow \infty} t_n$. If $\lim_{n \rightarrow \infty} s_n = +\infty$, then by **(a)**, we see that $\lim_{n \rightarrow \infty} t_n = \infty$, A contradiction. if $\lim_{n \rightarrow \infty} t_n = -\infty$, then $\lim_{n \rightarrow \infty} s_n = -\infty$, A contradiction. So we may assume that $-\infty < \lim_{n \rightarrow \infty} t_n < \lim_{n \rightarrow \infty} s_n < \infty$. Let

$$A = \lim_{n \rightarrow \infty} t_n, B = \lim_{n \rightarrow \infty} s_n.$$

Let $\epsilon = \frac{B-A}{4}$. For this $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$A - \epsilon < t_n < A + \epsilon = \frac{B + 3A}{4}.$$

Also for this $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$,

$$\frac{A + 3B}{4} = B - \epsilon < s_n < B + \epsilon.$$

Since $A < B$,

$$\frac{3A + B}{4} < \frac{B + 3A}{4}.$$

Therefore for $n \geq \max\{N_1, N_2\}$, $t_n < s_n$, A contradicton. □

10. P55. # 9.11

Proof. **(a).** Let

$$B = \inf\{t_n : n \in \mathbb{N}\}.$$

As an infimum of real numbers,

$$B < \infty.$$

So

$$-\infty < B < \infty.$$

Since $\lim_{n \rightarrow \infty} s_n = \infty$, for any $M > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$s_n > M - B.$$

Since $B = \inf\{t_n : n \in \mathbb{N}\}$, $t_n \geq B$. Therefore

$$s_n + t_n > M,$$

which implies that $\lim_{n \rightarrow \infty} (s_n + t_n) = \infty$.

(b). The case where $\lim_{n \rightarrow \infty} t_n < \infty$ is proven similarly as in **(a)**. We discuss the case where $\lim_{n \rightarrow \infty} t_n = \infty$. For $M > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$s_n > \frac{M}{2}, t_n > \frac{M}{2}.$$

So we have

$$s_n + t_n > \frac{M}{2} + \frac{M}{2} = M.$$

Therefore we have

$$\lim_{n \rightarrow \infty} (s_n + t_n) = +\infty.$$

The claim in **(c)** is proven similarly. □

11. P56. # 9.15

Proof. If $a = 0$, then $\frac{a^n}{n!} = 0$. So $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$.

If $a \in \mathbb{R}$ and $a \neq 0$, we prove that

$$\lim_{n \rightarrow \infty} \frac{a^n}{n!} = \lim_{n \rightarrow \infty} \frac{|a|^n}{n!} = 0.$$

We write

$$\frac{|a|^n}{n!} = \frac{|a| \times \cdots \times |a|}{1 \times 2 \times \cdots \times n} = |a| \times \frac{|a|}{2} \times \frac{|a|}{3} \times \cdots \times \frac{|a|}{n}.$$

Then

$$\frac{|a|^n}{n!} \leq |a|^{n-1} \frac{|a|}{n}.$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{|a|^n}{n!} = 0.$$

□

12. P56. # 9.18

Proof. The proof is skipped.

□

13. P64. # 10.1

Proof. In **(a)**, $\frac{1}{n}$ is a decreasing and bounded sequence.

In **(b)**, $\frac{(-1)^n}{n^2}$ is a bounded sequence.

In **(c)**, n^5 is an increasing sequence.

In **(d)**, $\sin \frac{n\pi}{7}$ is a bounded sequence.

In **(e)**, $(-2)^n$ is neither increasing nor decreasing; it is not bounded either.

In **(f)**, $\frac{n}{3^n}$ is a decreasing and bounded sequence.

□

14. P65. # 10.3

Proof. We recall that $s_n = K + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n}$ for all n . Since each d_j belongs to the set $\{0, 1, 2, \dots, 9\}$, we see that $0 \leq d_j \leq 9$. So

$$s_n \leq K + \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = K + 1 - \frac{1}{10^n} < K + 1.$$

□

15. P65. # 10.6

Proof. **(a)**. Since $\sum_{n=1}^{\infty} 2^{-n} = 1$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\sum_{n \geq N} \frac{1}{2^n} < \epsilon.$$

For $m > n \geq N$,

$$\begin{aligned} (3) \quad |s_m - s_n| &= |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n| \\ &\leq |s_m - s_{m-1}| + \cdots + |s_{n+1} - s_n| \\ &\leq \sum_{n \geq N} \frac{1}{2^n} < \epsilon. \end{aligned}$$

Therefore we prove that $\{s_n\}$ is a Cauchy sequence. Hence it is convergent.

(b). We assume that $s_n = \sum_{k=1}^n \frac{1}{k}$. Then $\{s_n\}$ satisfies the condition

$$|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}.$$

However $\sum_{k=1}^{\infty} \frac{1}{k}$ is a harmonic series.

□

16. P65. # 10.7

Proof. By the completeness Axiom, $\alpha = \sup S$ exists. For any $\frac{1}{n} > 0$, there exists $\alpha - \frac{1}{n}$ is not an upper bound, i.e., there exists $s_n \in S$ such that

$$\alpha - \frac{1}{n} < s_n \leq \alpha \leq \alpha + \frac{1}{n}.$$

By the squeezing theorem,

$$\lim_{n \rightarrow \infty} s_n = \alpha = \sup S.$$

□

17. P65. # 10.9

Proof. (a). $s_1 = 1, s_2 = \frac{1}{2}, s_3 = \frac{1}{6}, s_4 = \frac{1}{48}$.

(b). Firstly $s_n \geq 0$. Then we prove that $\{s_n\}_{n \geq 1}$ is a decreasing sequence. Then by the monotone convergence theorem, we see that $\lim_{n \rightarrow \infty} s_n$ exists.

Indeed,

$$\begin{aligned}
 s_{n+2} - s_{n+1} &= \frac{n+1}{n+2}s_{n+1}^2 - \frac{n}{n+1}s_n^2 \\
 &\leq \frac{n}{n+1}s_{n+1}^2 - \frac{n+1}{n+2}s_n^2 \\
 (4) \qquad &= \frac{n+1}{n+2}(s_{n+1}^2 - s_n^2) \\
 &= \frac{n+1}{n+2}(s_{n+1} + s_n)(s_{n+1} - s_n).
 \end{aligned}$$

Since $s_2 - s_1 \leq 0$, inductively we prove that $s_{n+1} \leq s_n$ for all n . Therefore $\{s_n\}_{n \geq 1}$ is decreasing.

(c). In (b), we have proved that $\alpha = \lim_{n \rightarrow \infty} s_n$ exists. Taking limits on both sides of $s_{n+1} = \frac{n}{n+1}s_n^2$, we see that

$$\alpha = \alpha^2,$$

which implies that

$$\alpha = 0, \alpha = 1.$$

Since $s_2 = \frac{1}{2}$ and $\{s_n\}_{n \geq 1}$ is a decreasing sequence, we see

$$\alpha = 0.$$

□

18. P66. # 10.12

Proof. (a). $t_n \geq 0$ for all n , and $t_{n+1} \leq t_n$ for all $n \in \mathbb{N}$. This proves that $\{t_n\}$ is a bounded and decreasing sequence. Therefore by the monotone convergence theorem, $\lim_{n \rightarrow \infty} t_n$ exists.

(b). From (c), $\lim_{n \rightarrow \infty} t_n = \frac{1}{2}$.

(c). From $t_{n+1} = \frac{n(n+2)}{(n+1)^2} = \frac{n+2}{n+1} \times \frac{n+1}{n}$, by math induction, we prove that $t_n = \frac{n+1}{2n}$.

Alternatively,

$$\begin{aligned}
 t_n &= \frac{n+1}{n} \times \frac{n-1}{n} \times \frac{n}{n-1} \times \frac{n-2}{n-1} \times \cdots \times t_2 \\
 (5) \qquad &= \frac{n+1}{n} \times \frac{n-1}{n} \times \frac{n}{n-1} \times \frac{n-2}{n-1} \times \cdots \times \frac{3}{2} \times \frac{1}{2} \\
 &= \frac{n+1}{2n}.
 \end{aligned}$$

(d). From (c), $\lim_{n \rightarrow \infty} t_n = \frac{1}{2}$.

□

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