HOMEWORK 3

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Abstract. Please send me an email if you find mistakes. Thanks.

1. P76. # 11.2

Proof. **(a).**

$$\begin{aligned} a_{2n} &= 1, \text{constant} \\ b_n &= \frac{1}{n}, \text{decreasing} \\ c_n &= n^2, \text{ increasing} \\ d_n &= \frac{6n+4}{7n-3}, \text{ decreasing} \end{aligned}$$

(b). For each subsequence above, the sub-sequential limits are correspondingly

$$\begin{aligned} &\{1, -1\}, \\ &\{0\}, \\ &\{\infty\}, \\ &\{\frac{6}{7}\}. \end{aligned}$$

(c). For each subsequence above, the lim sup and the lim inf are correspondingly

$$\begin{aligned} &1, -1, \\ &1, 0, \\ &\infty, 1, \\ &\frac{5}{2}, \frac{7}{6}. \end{aligned}$$

(d). The sequences $\{b_n\}, \{d_n\}$ converge. However, $\{c_n\}$ diverges to ∞.

(e). The sequences $\{a_n\}, \{b_n\}, \{d_n\}$ are bounded.

$\square$

2. P76. # 11.3

The proof is skipped.

3. P77. # 11.6

Proof. Suppose that $\{a_{n_k}\}_{k\geq 1}$ is a subsequence of $\{a_n\}$; $\{a_{n_{kj}}\}$ is a subsequence of a_{n_k}. Then the index n_{kj} satisfies,

$$n_{k1} < n_{k2} < \cdots < n_{kj} < \cdots$$

and also

$$n_{kj} \geq j.$$

From the definition of subsequences, $\{a_{n_{kj}}\}$ is a subsequence of a_n. $\square$

4. P78. # 11.11

Proof. If $\sup S$ is in S, then we denote $s_n = \sup S$ for all n. Then it is a constant sequence.

If $\sup S$ is not in S, then for $\epsilon = 1$, there exists $s_1 \in S$ such that

$$\sup S - 1 < s_1 < \sup S.$$

For $\epsilon = \min\{\sup S - s_1, \frac{1}{2}\}$, there exists $s_2 \in S$ such that

$$\sup S - \epsilon < s_2 < \sup S.$$

This proves

$$s_1 \leq \sup S - \epsilon < s_2, i.e., s_1 < s_2.$$

Supppose that $s_1, s_2, \cdots, s_n$ are chosen such that

$$s_1 < s_2 < \cdots < s_n,$$

and

$$\sup S - \frac{1}{j} < s_j < \sup S,\ 1 \leq j \leq n.$$

Then for $\epsilon = \min\{\sup S - s_n, \frac{1}{n+1}\}$, there exists s_{n+1} such that

$$\sup S - \epsilon < s_{n+1} < \sup S,$$

which yields,

$$s_n < s_{n+1},\ \sup S - \frac{1}{n+1} < s_{n+1} < \sup S.$$

Therefore we see that there exists an increasing sequence $\{s_n\}$ which converges to $\sup S$. □

5. P104. # 14.1

Proof. (a), (b),(c), converge by ratio test because the limits

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}, 0, \frac{1}{3}, \text{respectively.}$$

(d) diverges by ratio test because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty > 1.$$

(e) converges absolutely by the comparison test, as $|\frac{\cos^2 n}{n^2}| \le \frac{1}{n^2}$.

For (f), for $n \ge 10$,

$$\log n < n,\ \frac{1}{\log n} > \frac{1}{n}.$$

By the comparison test, the series diverges because the harmonic series diverges. □

6. P104. # 14.2

Proof. For (a), for $n \ge 10$,

$$\frac{n-1}{n^2} \frac{n/2}{n^2},\ \frac{n-1}{n^2} > \frac{1}{2n}.$$

By the comparison test, the series diverges because the harmonic series diverges.

(b) diverges because the limit of $\{a_n = (-1)^n\}$ does not converge to zero.

(c) converges by the comparison test as

$$\frac{3n}{n^3} \le \frac{3}{n^2}.$$

(d), (e), (g) converge by the ration test. The limits are,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3}, 0, \frac{1}{2}, \text{respectively.}$$

(f) converges by the root test as

$$\limsup \sqrt[n]{\frac{1}{n^n}} = 0 < 1.$$

□

7. P104. # 14.5

Proof. Denote the partial sums of $\sum a_n$ and $\sum b_n$ are

$$A_n = \sum_{k=1}^{n} a_n, B_n = \sum_{k=1}^{n} b_n.$$

It is known that

$$\lim_{n\to\infty} A_n = A, \lim_{n\to\infty} B_n = B.$$

Therefore

$$\lim_{n\to\infty} (A_n + B_n) = A + B.$$

This proves that

$$\sum_{n\to\infty} (a_n + b_n) = \sum a_n + \sum b_n.$$

The claim in **(b)** is proven similarly.

For (c), this is not a reasonable conjecture. In general, it fails. Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. Then

$$a_n b_n = \frac{1}{n}.$$

However $\sum_{n=1}^{\infty} 1n$ is a harmonic series and so diverges.

For $a_n = b_n = \frac{1}{n^2}$,

$$a_n b_n = \frac{1}{n^4} < \frac{1}{n^2}, \text{ for } n \geq 2.$$

Therefore in this case although both sides converge, $\sum a_n b_n < \sum a_n \sum b_n$.

□

8. P104. # 14.6

Proof. **(a).** $\{b_n\}$ is a bounded sequence, there exists $M > 0$ such that

$$|b_n| \leq M$$

for all n. Then

$$|a_n b_n| \leq M|a_n|.$$

Hence by the comparison test, the series $\sum a_n b_n$ converges absolutely.

(b). Let $b_n = 1$ for all n. Then Corollary 14.7 is a special case of this claim in **(a)**.

$\square$

9. P105. # 14.12

Proof. Let $b_N = \inf\{|a_n| : n \geq N\}$. Then
$$\liminf |a_n| = \lim_{N\to\infty} b_N = 0.$$

For $\epsilon = 1$, there exists K_1 such that for $N \geq K_1$,
$$b_N = \inf\{|a_n| : n \geq N\} \leq 1.$$
Fix some N. For $\epsilon = 1$, there exists $n_1 \geq N$,
$$|a_{n_1}| < b_N + 1 < 2.$$

For $\epsilon = \frac{1}{2}$, there exists $K_2 > n_1$ such that for $N \geq K_2$,
$$b_N = \inf\{|a_n| : n \geq N\} \leq \frac{1}{2}.$$
Fix some N. For $\epsilon = 1$, there exists $n_2 \geq N$,
$$|a_{n_2}| < b_N + 1 < 1.$$

Suppose that $a_{n_1}, \cdots, a_{n_k}$ are chosen such that
$$|a_{n_j}| \leq \frac{2}{2^j}$$
for $1 \leq j \leq k$, and
$$n_1 < n_2 < \cdots n_k.$$

For $\epsilon = \frac{1}{2^{k+1}}$, there exists K_{k+1} such that for $N \geq K_{k+1}$,
$$b_N = \inf\{|a_n| : n \geq N\} \leq \frac{1}{2^{k+1}}.$$
Fix some N. For $\epsilon = 1$, there exists $n_{k+1} \geq N$,
$$|a_{n_1}| < \frac{2}{2^{k+1}}.$$

Thus a subsequence a_{n_k} are chosen such that
$$|a_{n_k}| \leq \frac{2}{2^k}.$$
Therefore by the comparison test, $\sum_k a_{n_k}$ converges absolutely. $\square$

10. P105. #14.13

The proof follows from the hint, and so it is skipped.

11. P105. # 14.14

Proof. We observe that for $2^k \le n \le 2^{k+1} - 1$, where $k \in \mathbb{N}$,

$$a_n = \frac{1}{2^{k+1}}.$$

We denote the partial sums of $\sum \frac{1}{n}$ by s_n.

Therefore we have

$$\sum_{n=1}^{2^N - 1} \ge 1 + 2 \times \frac{1}{2^2} + 2^2 \times \frac{1}{2^3} + \cdots + 2^{N-1} \times \frac{1}{2^N} = \frac{N+1}{2}.$$

This proves that the partial sums $S_{2^N - 1}$ diverges. On other hand, for any $n \ge 2$, there exists N such that

$$2^N \le n \le 2^{N+1}.$$

So

$$s_n \ge s_{2^N} \ge s_{2^N - 1} > \frac{N+1}{2}.$$

This proves that s_n diverges, too. □

12. P109. # 15.1

Proof. (a) converges because of the alternating series test. (b) diverges because of the ratio test. □

13. P109. #15.2

Proof. (a) diverges: For $n = 6k$, then

$$\left(\sin \frac{n\pi}{6}\right)^n = (-1)^{k \times 6k} = 1.$$

(b) diverges by the same reason. □

14. P109. # 15.3

Proof. This follows from the integral test:

$$\int_{10}^{\infty} \frac{1}{x(\log x)^p} dx = \int_{\log 10}^{\infty} \frac{1}{t^p} dt < \infty.$$

□

15. P109. # 15.6

Proof. **(a).** $a_n = \frac{1}{n}$.

(b). If $a_n \geq 0$ and $\sum a_n$ is convergent, $\lim a_n = 0$. Then there exists $M > 0$ such that

$$a_n \leq M.$$

By the proof of Ex. 14.6, $\sum a_n^2$ is convergent.

(c).

$$a_n = \frac{(-1)^n}{\sqrt{n}}.$$

□

16. P109. #15.7

Proof. Firstly we prove that all $a_n \geq 0$. Otherwsie, if $a_N < 0$ for some N, then because $\{a_n\}$ is decreasing,

$$a_n \leq a_N$$

for all $n \geq N$. Therefore $\lim a_n \leq a_N < 0$. A contradiction.

By the Cauchy criterion, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m > n > N$,

$$|s_m - s_n| < \epsilon.$$

For $m = 2n$,

$$a_{n+1} + a_{n+2} + \cdots + a_{2n} < \epsilon.$$

This implies that

$$na_2 n < \epsilon.$$

Therefore $na_n < 2\epsilon$ for even n. Similarly we can prove that for odd n,

$$na_n < 2\epsilon.$$

Therefore $\lim_n na_n = 0$.

$\square$

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