HOMEWORK 4

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ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P130 . # 17.1

Proof. **(a).** The domain of the new functions should be the intersection of the old domains of the two functions, $f+g$ and fg; so it is $(-\infty, 4]$. For the composite functions, the domain of the inner function should give outputs in the domain of the outside function. So for $f \circ g$, the domain is $[-2, 2]$ and for $g \circ f$, the domain is $(-\infty, 4]$.

(b).

$$\begin{aligned} &f \circ g(0) = 2, \\ &g \circ f(0)4, \\ &f \circ g(1) = \sqrt{3}, \\ &g \circ f(0)3, \\ &f \circ g(2) = 0, \\ &g \circ f(0)2. \end{aligned}$$

(c). From (b), they are not equal.

(d). $f \circ g(3)$ does not make sense; however, $g \circ f(3) = 1$ make sense. □

2. P131. # 17.4

Proof. If $a > 0$, we prove that $\sqrt{x}$ is continuous at $x = a$. For any $\epsilon > 0$, we need to find $\delta > 0$ such that $|x-a| < \delta$ and $x > 0$,

$$|\sqrt{x} - \sqrt{a}| < \epsilon.$$

We know that

$$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \frac{|x-a|}{\sqrt{a}}.$$

We take $\delta = \sqrt{a}\epsilon$.

For $a = 0$, for any $\epsilon > 0$, we take $\delta = \epsilon^2$, then for $|x| < \delta$,

$$|\sqrt{x} - \sqrt{0}| = \sqrt{x} \leq \sqrt{\delta} = \epsilon.$$

This proves that $\sqrt{x}$ is continuous at $x = 0$. So $\sqrt{x}$ is continuous at all $x \geq 0$. $\square$

3. P132. # 17.9(a)

Proof. For any $\epsilon > 0$, we need to find $\delta > 0$ such that for $|x - 2| < \delta$,

$$|x^2 - 2^2| < \epsilon.$$

We see that

$$|x^2 - 2^2| = |x + 2||x - 2|.$$

So firstly we take $\delta < 1$, so $1 < x < 3$. so $3 < |x + 2| = x + 2 < 5$. Then we take $\delta < \frac{\epsilon}{5}$, then we have

$$|x^2 - 2^2| < 5|x - 2| < \epsilon.$$

So finally we take $0 < \delta < \min\{1, \frac{\epsilon}{5}\}$.

$\square$

4. P132. # 17.10 (b)

Proof. **(b).** We take $x_n = \frac{1}{2n\pi}$, then $x_n \to 0$ as $n \to \infty$. However

$$\sin \frac{1}{x_n} = 0.$$

We take another sequence, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, then we know that $y_n \to 0$.

$$\sin \frac{1}{y_n} = 1.$$

Thus we find two sequences both converging to zero. But their limits are 0 and 1, respectively. This proves that $g(x)$ is not continuous at zero. $\square$

5. P132. # 17.11

Proof. From Theorem 17.2, if f is continuous at x_0, then for any monotonic sequence x_n in $\mathrm{dom}(f)$ converging to x_0, $f(x_n) = f(x_0)$.

Conversely, for any sequence y_n converging to x_0, we need to prove that

$$f(y_n) \to f(x_0), \text{ as } n \to \infty.$$

We prove it by contradiction. Suppose that it fails. Then there exists a subsequence y_n such that the convergence fails. There exists $\epsilon_0 > 0$, for any $\frac{1}{n}$, there exists y_{n_k} such that

$$|f(y_{n_k}) - f(x_0)| \geq \epsilon_0. \tag{1}$$

However for y_{n_k}, there exists a subsequence $y_{n_{k_j}}$ such that $y_{n_{k_j}}$ is monotone. Therefore by the hypothesis,

$$\lim_{j\to\infty} f(y_{n_{k_j}}) = f(x_0).$$

This is a contradiction to (1). □

6. P132. # 17.13 (A)

Proof. We note that both rationals and irrationals are dense in the real numbers. For any real number a, there exists rational sequence $x_n \to a$ and irrational sequence $y_n \to a$. However

$$f(x_n) = 1, f(y_n) = 0.$$

This proves that f is not continuous at a. □

7. P138. # 18.1

Proof. This is obvious. □

8. P138. # 18.2

Proof. A subsequence may converges to an endpoint of (a, b). For instance, $f(x) = \frac{1}{x}$ on $(0, 1)$. We take the subsequence $x_n = \frac{1}{n}$. □

9. P139. # 18.4

Proof. We construct the function as follows: $f(x) = \frac{1}{x-x_0}$. Since there exists a sequence x_n in S converging to x_0, then there either exists a subsequence of $\{x_n\}$ converging to x_0 either from the left hand side or from the right hand side. If it is from the left hand, then f is not bounded above. If it is from the right hand, then f is not bounded below. □

10. P139. # 18.5

Proof. **(a).** We consider the function $h(x) = f(x) - g(x)$. Then

$$h(a) = f(a) - g(a) \geq 0, h(b) = f(b) - g(b) \leq 0,$$

so for 0, by the intermediate value theorem, we see that there exists $x_0 \in [a, b]$ such that

$$h(x_0) = 0, \text{ i.e. } f(x_0) = g(x_0).$$

(b). Let $g(x) = x$.

□

11. P139. # 18.8

Proof. Since $f(a)f(b) < 0$, then $f(a)$ and $f(b)$ have different signs. So for 0, by the intermediate value theorem, there exists x_0 between a and b such that

$$f(x_0) = 0.$$

□

12. P139. # 18.12

Proof. **(a).** This is done in Exercise # 17.10 (b).

(b). We observe that f is continuous on the real line except for 0. It has the intermediate value property on either the positive real axis or the negative real axis. Suppose y is between $f(a)$ and $f(b)$. I

Case 1. If a, b have the same signs, we apply the intermediate value theorem on one side.

Case 2. If a, b have different signs. Suppose that $a < 0 < b$ and $0 < -a \leq b$. We first assume that $[a, b] \subset [-\frac{1}{2\pi}, \frac{1}{2\pi}]$. There exists $n \in \mathbb{N}$ such that

$$-\frac{1}{2n\pi} \leq a \leq -\frac{1}{2(n+1)\pi}.$$

Then we consider $\frac{1}{a_0} = -\frac{1}{a} + \pi$, which is obtained by reflecting a about the origin and translating it π. Then

$$a_0 = \frac{1}{-\frac{1}{a} + \pi} < \frac{1}{-\frac{1}{a}} = -a,$$

and

$$\sin\frac{1}{a_0} = \sin\frac{1}{a},$$

and a_0 and b are on the same side to the origin. Then we can apply the intermediate value theorem on one side.

Secondly if $a < -\frac{1}{2\pi}$ or $b > \frac{1}{2\pi}$, there exists $a_1, b_1 \in [-\frac{1}{2\pi}, \frac{1}{2\pi}]$ such that

$$\sin\frac{1}{a_1} = \sin\frac{1}{a}, \sin\frac{1}{b_1} = \sin\frac{1}{b}.$$

Indeed, for $a < -\frac{1}{2\pi}$, we see that

$$-4\pi < \frac{1}{a} - 2\pi < 2\pi.$$

Setting $a_1 = \frac{1}{\frac{1}{a} - 2\pi}$. Then

$$-\frac{1}{2\pi} < a_1 < 0.$$

Then it reduce to the situation considered above. □

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