HOMEWORK 5

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ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P151. # 19.1

Proof. **(a). (b).** The function is uniformly continuous on $[0,\pi]$ by Theorem 19.2 because it is continuous on $[0,\pi]$.

(c). This function is uniformly continuous on $(0,1)$ by Theorem 19.5 because it can be extended to be a continuous function on $[0,1]$. The extension still takes the same form as x^3.

(d). f is not uniformly continuous on $\mathbb{R}$: We choose $x_n = n + \frac{1}{n}$ and $y_n = n$, then

$$|x_n - y_n| = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

However,

$$x_n^3 - y_n^3 = 3n(n + \frac{1}{n}) \times \frac{1}{n} \geq n.$$

(e). f is not uniformly continuous on $(0,1]$: We choose $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$, then

$$|x_n - y_n| = \frac{1}{2n} \to 0 \text{ as } n \to \infty.$$

However,

$$|x_n^3 - y_n^3| = |n^3 - 8n^3| = 7n^3.$$

(f). f is not uniformly continuous on $(0,1]$ because f can not extend to a continuous function on $[0,1]$. For $x_n = \frac{1}{\sqrt{2n\pi}}$ and $y_n = \frac{1}{\sqrt{2n\pi + \pi/2}}$,

$$f(x_n) \to 0, \text{ but } f(y_n) \to 1.$$

(g). f is a uniformly continuous function on $(0,1]$ because f can be extended to be a continuous function on $[0,1]$. Define g to be

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{for } (0,1], \\ 0, & \text{for } x = 0. \end{cases}$$

The function g is continuous on $(0,1]$; however it is also continuous at $x = 0$ because

$$0 \leq |x^2 \sin \frac{1}{x}| \leq x^2.$$

So g is a continuous function on $[0,1]$. □

2. P151. # 19.2

Proof. **(a).** For $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{3}$ such that for $|x-y| < \delta$,

$$|f(x) - f(y)| = 3|x-y| < 3\delta = \epsilon.$$

This proves that f is a uniformly continuous function on $\mathbb{R}$.

(b). For $f(x) = x^2$ on $[0,3]$: for any $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{6}$, such that for $|x-y| < \delta$,

$$|x^2 - y^2| = (x+y)|x-y| < 6|x-y|$$

because $0 \leq x, y \leq 3$. Then

$$|x^2 - y^2| < 6|x-y| = \epsilon.$$

So f is a uniformly continuous function on $[0,3]$. □

3. P151. # 19.3

Proof. **(a).** For any $\epsilon > 0$, there exists $\delta = \epsilon$, such that for any $|x-y| < \delta$,

$$|f(x) - f(y)| = |\frac{x}{x+1} - \frac{y}{y+1}| = \frac{|x-y|}{(x+1)(y+1)} \leq |x-y|$$

because $0 \leq x, y \leq 2$. so

$$|f(x) - f(y)| \leq \epsilon.$$

This proves that f is a uniformly continuous function on $[0,2]$. □

4. P151. # 19.4

Proof. **(a).** We prove it by contradiction. Suppose that f is not bounded. Then f may not be bounded above or f may not be bounded below. Suppose that f is not bounded above. For any $n \in \mathbb{N}$, there exists $x_n \in S$,

$$f(x_n) > n.$$

Since $\{x_n\}$ is bounded, by the Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of x_n such that x_{n_k} converges. Then $\{x_{n_k}\}$ is a Cauchy. Therefore $f(x_{n_k})$ is Cauchy, too, because f is a uniformly continuous function. Therefore $\lim_{k\to\infty} f(x_{n_k})$ exists. So $\{f(x_{n_k})\}$ is bounded. However by assumption,

$$f(x_{n_k}) \geq n_k, \text{ and } n_k \to \infty, \text{as } k \to \infty.$$

which proves that $f(x_{n_k})$ is not bounded. A contradiction.

(b). If it is uniformly continuous, then f is bounded on $(0,1)$. However, for $x_n = \frac{1}{n}$,

$$f(x_n) = \frac{1}{x_n^2} = n^2 \to \infty.$$

It is not bounded. Therefore it is not uniformly continuous.

□

5. P152. # 19.6

Proof. **(a).** We compute $f'(x) = \frac{1}{2\sqrt{x}}$. Then $f'(x)$ is not bounded on $(0,1]$. However f is uniformly continuous on $(0,1]$. We observe that for $x, y \in (0,1]$,

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}.$$

This can be proven, if $x \geq y$,

$$\sqrt{x} \leq \sqrt{x-y} + \sqrt{y}.$$

and if $x \leq y$,

$$\sqrt{y} \leq \sqrt{y-x} + \sqrt{x}.$$

Then the inequality holds. For any $\epsilon > 0$, we take $\delta = \epsilon^2$, then for $|x-y| \leq \delta$,

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|} \leq \epsilon.$$

This proves that $\sqrt{x}$ is uniformly continuous on $(0,1]$.

(b). The proof in part (a) applies in this case, too. □

6. P152. # 19.7

Proof. **(a).** If $k = 0$, then the claim is proven. Suppose that $k > 0$. We know that f is uniformly continuous on $[k, \infty)$: for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that for $|x - y| < \delta_1$,

$$|f(x) - f(y)| < \epsilon.$$

Also f is continuous on $[0, 2k]$ and so is uniformly continuous on $[0, 2k]$: for the same $\epsilon > 0$, there exists $\delta_2 > 0$ such that for $|x - y| < \delta_2$,

$$|f(x) - f(y)| < \epsilon.$$

Then we claim that f is uniformly continuous on $[0, \infty)$. For the same $\epsilon > 0$, we take $\delta < \min\{k, \delta_1, \delta_2\}$, then if $|x - y| < \delta$,

$$|f(x) - f(y)| < \epsilon.$$

This is because $|x - y| < \delta$ implies two cases, if $x \in [0, k]$, then since $\delta \leq k$, $x, y \in [0, 2k]$. We can apply uniform continuity of f on $[0, 2k]$. If $x \in [2k, \infty)$, then since $\delta \leq k$, $x, y \in [k, \infty)$. Then we can apply the uniform continuity of f on $[k, \infty)$. If $x \in (k, 2k)$, then we since $|x - y| < k$, then either x, y are in $[0, 2k]$ or in $[k, \infty)$. Therefore f is uniformly continuous on $[k, \infty)$.

(b). By Ex. 19.6 and part (a), $f = \sqrt{x}$ is uniformly continuous on $[0, \infty)$. □

7. P152. #19.9

Proof. **(a).** f is continuous when $x \neq 0$. When $x = 0$,

$$|x \sin \frac{1}{x}| \leq |x|.$$

So it is also continuous at $x = 0$. So it is continuous on $\mathbb{R}$.

(b). Yes. f is uniformly continuous on any bounded subset of $\mathbb{R}$. Indeed, any bounded subset is contained in a closed and bounded interval, $[-M, M]$, for some $M > 0$. Since f is uniformly continuous on $[-M, M]$, f is uniformly continuous on the bounded subset.

(c). Yes. Firstly we prove that

$$|\sin x - \sin y| \leq |x - y|.$$

This is easily proven by using the mean value theorem, since $\sin' x = \cos x$ and $|\cos x| \leq 1$.

Secondly, we prove that f is uniformly continuous on the interval $|x| \geq 10$. We write

$$|x \sin \frac{1}{x} - y \sin \frac{1}{y}| = |(x-y) \sin \frac{1}{x} - y(\sin \frac{1}{x} - \sin \frac{1}{y})| \leq |x-y| + |y| \frac{|x-y|}{|x||y|} \leq \frac{11}{10}|x-y|.$$

For any $\epsilon > 0$, there exists $\delta < \frac{1}{2}\epsilon$ such that for any $|x-y| < \delta$,

$$|f(x) - f(y)| \leq \frac{11}{10}|x-y| < \epsilon.$$

Therefore f is uniformly continuous on $|x| \geq 10$. By part (b), f is uniformly continuous on $[-10, 10]$. Hence by Ex.19.7, f is uniformly continuous on $\mathbb{R}$. □

8. P162. # 20.1

Proof.

$$\lim_{x\to\infty} f(x) = 1; \lim_{x\to 0+} f(x) = 1; \lim_{x\to 0-} f(x) = -1; \lim_{x\to -\infty} f(x) = 1.$$

However $\lim_{x\to 0} f(x)$ does not exist because the left-hand limit and the right hand limit are not equal. □

9. P162. # 20.2

Proof.

$$\lim_{x\to\infty} f(x) = \infty; \lim_{x\to 0+} f(x) = 0; \lim_{x\to 0-} f(x) = 0; \lim_{x\to -\infty} f(x) = -\infty.$$

However $\lim_{x\to 0} f(x) = 0$ because the left-hand limit and the right hand limit exist and are equal. □

10. P162. # 20.5

Proof. We can rewrite f,

$$f(x) = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases}$$

Therefore the limits in Ex. 20.1 hold. □

11. P162. #20.6

Proof. Since $f(x) = \frac{x^3}{|x|}$, we see that

$$|f(x)| \leq x^2.$$

So

$$\lim_{x\to 0+} f(x) = 0; \lim_{x\to 0-} f(x) = 0.$$

However, f also satisfies

$$|f(x)| = x^2.$$

This proves that

$$\lim_{x\to\infty} f(x) = \infty; \lim_{x\to -\infty} f(x) = -\infty.$$

□

12. P162. # 20.11

Proof. **(a).**

$$\lim_{x\to a} \frac{x^2 - a^2}{x - a} = \lim_{x\to a} \frac{(x+a)(x-a)}{x-a} = \lim_{x\to a}(x+a) = 2a.$$

(b). Since $x - b = (\sqrt{x} + \sqrt{b})(\sqrt{x} - \sqrt{b})$,

$$\lim_{x\to b} \frac{\sqrt{x} - \sqrt{b}}{x - b} = \lim_{x\to b} \frac{1}{\sqrt{x} + \sqrt{b}} = \frac{1}{2\sqrt{b}}.$$

Similarly in (c), $\lim_{x\to a} \frac{x^3 - a^3}{x-a} = 3a^2$. □

13. P163. # 20.12

Proof. **(a).**

$$\lim_{x\to 2+} f(x) = \infty; \lim_{x\to 2-} f(x) = \infty; \lim_{x\to 1+} f(x) = \infty; \lim_{x\to 1-} f(x) = -\infty.$$

(b). They do not exist. Even though at 2,

$$\lim_{x\to 2+} f(x) = \infty = \lim_{x\to 2-} f(x),$$

the limit does not exist at 2 since the limit value is ∞. □

14. P163. # 20.16

Proof. **(a).** We prove it by contradiction. Suppose that $L_1 > L_2$. Take $\epsilon = \frac{L_1 - L_2}{2}$. For this ϵ, $\lim_{x \to a+} f_1(x) = L_1$ implies that there exists $\delta_1 < \frac{b-a}{2}$, such that for $a < x < a + \delta_1 < b$,

$$|f_1(x) - L_1| < \epsilon.$$

This implies

$$f_1(x) > L_1 - \epsilon = \frac{L_1 + L_2}{2}, \text{for } a < x < a + \delta_1.$$

The limit $\lim_{x \to a+} f_2(x) = L_2$ implies that there exists $\delta_2 < \frac{b-a}{2}$ such that for $a < x < a + \delta_2 < b$,

$$|f_2(x) - L_2| < \epsilon.$$

This implies,

$$f_2(x) < L_2 + \epsilon = \frac{L_1 + L_2}{2}, \text{ for } a < x < a + \delta_2.$$

Therefore for $a < x < a + \min\{\delta_1, \delta_2\}$,

$$f_2(x) < \frac{L_1 + L_2}{2} < f_1(x).$$

A contradiction. This proves that $L_1 \leq L_2$.

(b). No. An example, $f_1(x) = \sin x$ and $f_2(x) = x$ for $x \in (0, \pi/2)$. Then we know that $\sin x < x$. But

$$\lim_{x \to 0+} \sin x = 0; \lim_{x \to 0+} x = 0.$$

□

15. P163. # 20.17

Proof. Given $\epsilon > 0$. The limit $\lim_{x \to a+} f_1(x) = L$ implies that there exists $0 < \delta_1 < a + \frac{b-a}{2}$, such that for $a < x < a + \delta_1 < b$,

$$|f_1(x) - L_1| < \epsilon.$$

This implies

$$L - \epsilon < f_1(x) < L + \epsilon, , \text{for } a < x < a + \delta_1.$$

The limit $\lim_{x \to a+} f_3(x) = L$ implies that there exists $0 < \delta_1 < a + \frac{b-a}{2}$ that for $a < x < a + \delta_2 < b$,

$$|f_3(x) - L| < \epsilon.$$

This implies,

$$L - \epsilon < f_3(x) < L + \epsilon, \text{ for } a < x < a + \delta_2.$$

Therefore for $a < x < a + \min\{\delta_1, \delta_2\}$,

$$L - \epsilon < f_1(x) \leq f_2(x) \leq f_3(x) < L + \epsilon,$$

i.e., $|f_2(x) - L| < \epsilon$. This proves that

$$\lim_{x \to a+} f_2(x) = L.$$

□

16. P163. # 20.20

Proof. **(a).** We first discuss the case where $-\infty < L_2 < \infty$. The limit $\lim_{x \to a^S} f_1(x) = \infty$ implies, for any $M > 0$, there exists $\delta_1 > 0$ such that for $0 < |x - a| < \delta_1$ and $x \in S$,

$$f_1(x) > M - (L_2 - 1).$$

Since $\lim_{x \to a^S} f_2(x) = L_2$, for $\epsilon = 1$, there exists $\delta_2 > 0$ such that for $0 < |x - a| < \delta_2$ and $x \in S$,

$$|f_2(x) - L_2| < \epsilon,$$

i.e.,

$$L_2 - 1 < f_2(x) < L_2 + 1.$$

Then for $0 < |x - a| < \min\{\delta_1, \delta_2\}$ and $x \in S$,

$$f_1(x) + f_2(x) > M - (L_2 - 1) + (L_2 - 1) = M.$$

This proves that $\lim_{x \to a^S} f_1(x) + f_2(x) = \infty$. The case where $L_2 = \infty$ is similar.

(b). We first discuss the case where $-\infty < L_2 < \infty$.

Since $\lim_{x \to a^S} f_2(x) = L_2$, for $\epsilon = \frac{L_2}{2}$, there exists $\delta_2 > 0$ such that for $0 < |x - a| < \delta_2$ and $x \in S$,

$$|f_2(x) - L_2| < \epsilon,$$

i.e.,

$$\frac{L_2}{2} < f_2(x) < \frac{3L_2}{2}.$$

The limit $\lim_{x \to a^S} f_1(x) = \infty$ implies, for any $M > 0$, there exists $\delta_1 > 0$ such that for $0 < |x - a| < \delta_1$ and $x \in S$,

$$f_1(x) > \frac{M}{L_2/2}.$$

Then for $0 < |x-a| < \min\{\delta_1, \delta_2\}$ and $x \in S$,

$$f_1(x)f_2(x) > \frac{M}{L_2/2} \times \frac{L_2}{2} = M.$$

This proves that $\lim_{x \to a^S} f_1(x)f_2(x) = \infty$. The case where $L_2 = \infty$ is similar.

(c). This is similar to part (b).

(d). It can be any positive real number. Let $a > 0 \in \mathbb{R}$.

$$\lim_{x \to 1+} \frac{a}{x-1} = \infty; \lim_{x \to 1+} x - 1 = 0.$$

But

$$\lim_{x \to 1+} \frac{a}{x-1} \times (x-1) = a.$$

□

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