HOMEWORK 9

SHUANGLIN SHAO

ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P289. # 33.1

Proof. The proof is similar to Theorem 33.1. So we skip it. □

2. P289. # 33.2

Proof. We show that $\sup\{cS\} = c\sup S$ for $c > 0$. The proof for infimum is similar. Firstly for $s \in S$,

$$cs \leq c\sup S.$$

So $c\sup S$ is an upper bound. For $\epsilon > 0$, there exists $s_0 \in S$ such that

$$s_0 \geq \sup S - \epsilon/c.$$

So

$$cs_0 \geq c\sup S - \epsilon.$$

This proves that $s\sup S$ is the least one among the upper bounds. Therefore

$$c\sup S = \sup(cS).$$

□

3. P289. # 33.4

Proof.

$$f = \begin{cases} 1, & \text{for } x rational numbers in [0,1], \\ -1, & \text{for } x irrational numbers in [0,1]. \end{cases}$$

One can compute the lower integral $L(f) = -1$ and $U(f) = 1$ as in the book. So f is not integrable. However,

$$|f = 1|$$

is a constant function on $[0,1]$ and so is integrable. □

4. P289. # 33.5

Proof. Here we use

$$\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)dx \ \leq \int_{-2\pi}^{2\pi} \ x^2 \sin^8(e^x) \ dx \leq \int_{-2\pi}^{2\pi} x^2 dx = \frac{16\pi^3}{3}.$$

□

5. P289. # 33.7

Proof. **(a).** For any partition$P = \{t_0 = a < t_1 < t_2 < \cdots < t_n = b\}$ and any $\epsilon > 0$, there exist x_k, y_k such that

$$M(f^2, [t_{k-1}, t_k]) - \epsilon < f^2(x_k)$$

and

$$f^2(y_k) \leq m(f^2, [t_{k-1}, t_k]) + \epsilon.$$

Then

$M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \leq f^2(x_k) - f^2(y_k) + 2\epsilon = (f(x_k) + f(y_k))(f(x_k) - f(y_k)) + 2\epsilon.$

Then

$$\begin{aligned} &M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \\ &\leq |(f(x_k) + f(y_k)| \, |f(x_k) - f(y_k)| + 2\epsilon \\ &\leq 2B|f(x_k) - f(y_k)| + 2\epsilon \leq 2B \left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \right) + 2\epsilon. \end{aligned}$$

Therefore

$M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \leq 2B \left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \right) + 2\epsilon.$

Since $\epsilon > 0$ is arbitrary,

$M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \leq 2B \left(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \right).$

This implies

$$U(f^2, P) - L(f^2, P) \leq 2B \left(U(f, P) - L(f, P) \right).$$

(b). f is integrable on $[a, b]$: for any $\epsilon > 0$, there exists P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon/2B.$$

So

$$U(f^2, P) - L(f^2, P) < \epsilon.$$

This proves that f^2 is integrable on $[a, b]$. □

Department of Mathematics, KU, Lawrence, KS 66045

E-mail address: `slshao@math.ku.edu`