HOMEWORK 1

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1. P 22. Ex. 1.3.1

a). $E = \{-3, 1\}$. So

$$\sup E = 1, \quad \inf E = -3.$$

b). $E = \{x, \quad 0 < x < 1.5\}$. So

$$\sup E = 1.5, \quad \inf E = 0.$$

c). $E = \{r = p/q \in \mathbb{Q} : r^2 < 5, r > 0\} = \{r \in \mathbb{Q} : 0 < r < \sqrt{5}\}$. So

$$\sup E = \sqrt{5}, \quad \inf E = 0.$$

d).

$$\sup E = 1.5, \quad \inf E = 0.$$

e).

$$\sup E = 1.5, \quad \inf E = -1.$$

f).

$$\sup E = 3, \quad \inf E = 7/4.$$

2. P22. Ex. 1.3.3

Proof. We follow the proof of Theorem 1.18. To begin with, we may assume that $0 < a < b$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that

$$a > \frac{\sqrt{2}}{n}, \text{ and } b - a > \frac{\sqrt{2}}{n}.$$

Set

$$E = \{k \in \mathbb{N} : \quad \frac{k\sqrt{2}}{n} \leq a\}.$$

We know that $E \neq \emptyset$, and E is bounded above. By the completeness axiom, $\sup E$ exists; moreover $\sup E \in E$ because E consists of integers.

Set $k_0 = \sup E$. Since k_0 is the supremum of E, $k_0 + 1 \notin E$. Hence

$$a < \frac{(k_0+1)\sqrt{2}}{n}.$$

From this and $b - a > \frac{\sqrt{2}}{n}$, we deduce that there exists $\xi = \frac{(k_0+1)\sqrt{2}}{n} \in \mathbb{R} \setminus Q$ such that $a < \xi < b$. □

3. P22. Ex. 1.3.6

Proof. Consider $-E = \{-x, \quad x \in E\}$. By the reflection principle, $\sup(-E)$ exists. For any $\varepsilon > 0$, there exists $a \in E$ such that

$$\sup(-E) - \varepsilon < -a \leq \sup(-E).$$

Then by the reflection principle again,

$$\inf E + \varepsilon > a \geq \inf E.$$

This finishes the proof of part (a).

The proof of part (b) is similar. □

4. P 23. Ex. 1.3.11

Proof. We may assume that $0 < a < b$. Consider $E = \{k \in \mathbb{N} : \quad k \leq b\}$. $E \neq \emptyset$ and E is bounded above. By the completeness axiom, there exists $k_0 = \sup E$; moreover $k_0 \in E$.

We first observe that $k_0 \leq b$. If $k_0 = b$, then we let $k = k_0 - 1$. Then since $b - a > 1$,

$$a < k < b.$$

On the other hand, if $k_0 < b$, we should have $b - k_0 < 1$ otherwise $k_0 + 1$ will be the supremum of E. In this case, let $k = k_0$, we have

$$a < k < b.$$

□

5. P 40. Ex. 1.6.1

Proof. Let $E = \{1, 3, 5, 7, \cdots,\}$ be the set of odd integers and $\mathbb{N}$ be the set of natural numbers. We define

$$f : E \to \mathbb{N} \text{ by } f(n) = 2n - 1.$$

It is not hard to verify that f is a bijection from E to $\mathbb{N}$. Then the set E is countable. □

6. P 40. Ex. 1.6.2

Proof. By Theorem 1.42, $\mathbb{Q}^3$ is at most countable; since $\mathbb{Q}$ is not finite, $\mathbb{Q}$ is countable. □

7. P 40. Ex. 1.6.3

Proof. Suppose that A is countable. Then there exists a bijection from $\mathbb{N}$ to A, which we denote by f. Let g be the function which maps A onto B. Then $g \circ f$ is an onto map from $\mathbb{N}$ to B. By theorem 1.40, B is at most countable. This is a contradiction. So A is uncountable. □

8. P 40. Ex. 1.6.6

Proof. **a).** **" $\Rightarrow$ ".** Let $E = \{\phi(k) : 1 \leq k \leq n, k \in \mathbb{N}\}$. Obviously $E \subset \{1, 2, \cdots, n\}$ and E contains n distinct elements as ϕ is $1-1$. This shows that $E = \{1, 2, \cdots, n\}$, which shows that ϕ is onto.

" $\Leftarrow$ ". If ϕ is onto, $E = \{1, 2, \cdots, n\}$. The set E consists of n distinct elements. So ϕ is $1-1$.

b). Let the bijection map $\psi : \{1, 2, \cdots, n\} \to A$. We consider

$$\{1, 2, \cdots, n\} \to_{\psi} A \to_{f} A \to \psi^{-1}\{1, 2, \cdots, n\}.$$

We use the part (a) to establish the claim in part (b). We omit the details. □

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