HOMEWORK 6

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1. Section 4.1. # 2.

Proof. By using the solution formula,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

Let $t = 0$. Then

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

By the expansion of ϕ,

$$1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \quad = \sin \frac{\pi x}{l} + A_2 \sin \frac{2\pi x}{l} + A_3 \sin \frac{3\pi x}{l} + \cdots + A_n \sin \frac{n\pi x}{l} + \cdots$$

Therefore

$$A_{2n} = 0, \text{ for } n = 1, 2, \cdots$$

and

$$A_{2n-1} = \frac{4}{\pi} \frac{1}{2n-1} = \frac{4}{(2n-1)\pi}, \text{ for } n = 1, 2, \cdots$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} e^{-\frac{(2n-1)^2\pi^2}{l^2}kt} \sin \frac{(2n-1)\pi x}{l}.$$

□

2. Section 4.1. # 4.

Proof. Let

$$u(x,t) = X(x)T(t).$$

Then

$$\big(T''(t) + rT'(t)\big) X(x) = c^2 X''(x) T(t).$$

Hence

$$-\frac{T''(t) + rT'(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda.$$

Since $X(0) = X(l) = 0$. So $\lambda > 0$. Let $\lambda = \beta^2$ with $\beta > 0$. Then we have the equation

$$X''(x) = -\lambda X(x) = -\beta^2 X(x)$$

which implies that

$$X(x) = A\cos\beta + B\sin\beta x.$$

The boundary conditions $u(0,t) = 0 = u(l,t)$ yields

$$A = 0, B \neq 0,\ \beta_n = \frac{n\pi}{l}, n = 1, 2, \cdots$$

Therefore

$$X_n(x) = \sin\frac{n\pi x}{l}.$$

We consider the equation

$$T''(t) + rT'(t) + c^2\beta_n^2 = 0. \tag{1}$$

Since $0 < r < \frac{\pi c}{l}$,

$$r^2 - \frac{4n^2c^2\pi^2}{l^2} < 0, \text{ for any } n = 1, 2, \cdots \tag{2}$$

Thus the quadratic equation $x^2 + rx + c^2\beta_n = 0$ has no real roots. Let λ_n be the non-real root in the complex number, then $r - \lambda_n$ is another root, which is different from λ_n because of (2). Hence the solution to (1) is

$$T_n(t) = A_n e^{\lambda_n t} + B_n e^{(r-\lambda_n)t}.$$

To conclude, we have

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n e^{\lambda_n t} + B_n e^{(r-\lambda_n)t}\right)\sin\frac{n\pi x}{l}.$$

This u is a solution to the system of equations provided that ϕ and ψ have the following expansion

$$\phi(x) = \sum_{n=1}^{\infty}(A_n + B_n)\sin\frac{n\pi x}{l},$$
$$\psi(x) = \sum_{n=1}^{\infty}(A_n\lambda_n + B_n(r-\lambda_n))\sin\frac{n\pi x}{l}.$$

□

3. Section 4.1. # 6.

Proof. Let $u(x,t) = X(x)T(t)$. Then

$$tX(x)T'(t) = X''(x)T(t) + 2X(x)T(t).$$

This implies

$$-\frac{tT'(t)}{T(t)} = -\frac{X''(x) + 2X(x)}{X(x)} = \lambda = \text{constant}.$$

Since $X(0,t) = X(\pi,t) = 0$, the equation

$$X''(x) = -(2+\lambda)X(x) \tag{3}$$

gives

$$-(2+\lambda) > 0.$$

Let $-(2+\lambda) = \beta^2$ with $\beta > 0$. Then solving (3) with the boundary conditions yields that

$$\beta_n = n,\ \lambda_n = -2 - n^2,\ X_n(x) = \sin nx,\ n = 1, 2, \cdots$$

Returning to the equation of t:

$$\frac{tT'(t)}{T(t)} = 2 + n^2.$$

We have

$$(\log T(t))' = \frac{2+n^2}{t}.$$

Integrating from 1 to t,

$$T(t) = A_n t^{n^2+2}.$$

Therefore

$$u(x,t) = \sum_n A_n t^{n^2+2} \sin nx.$$

Any finite combination of n satisfies that $u(x,0) = 0$. So there are an infinite number of solutions that satisfy the initial condition $u(x,0) = 0$. So uniqueness fails for this equation.

□

4. Section 4.2. # 1.

We formulae a lemma by following the same analysis as in the end of Section 4.1.

4.1. **Lemma.** *Suppose that* $-X'' = \lambda X$ *with the boundary conditions* $X(0) = 0, X'(l) = 0$. *Then*

$$\lambda > 0.$$

This is true under different boundary conditions $X'(0) = 0$ *and* $X(l) = 0$.

Proof. **Case 1.** Suppose that $\lambda = \beta^2$ with $\beta > 0$. Solving

$$-X''(x) = \lambda X(x),$$

we obtain

$$X(x) = C \cos \beta x + D \sin \beta x.$$

Thus

$$X'(x) = -C\beta \sin \beta x + D\beta \cos \beta x.$$

From the boundary conditions, we have

$$X(0) = C = 0, \text{ and } X'(l) = -C\beta \sin \beta l + D\beta \cos \beta l = 0.$$

Therefore

$$D\beta \cos \beta l = 0.$$

This implies that

$$\cos \beta l = 0.$$

Hence

$$\beta l = n\pi + \frac{\pi}{2}, \, n = 1, 2, \cdots$$

Case 2. Suppose that $\lambda = 0$. We have

$$X(x) = Cx + D.$$

Therefore from the boundary conditions $X(0) = 0$ and $X'(l) = 0$, we have

$$C = D = 0.$$

This leads to a trivial solution $u(x, t) = 0$.

Case 3. Suppose that $\lambda < 0$ or λ is a complex number with a nonzero imaginary part. The equation $-X'' = \lambda X$ leads to that the characteristic polynomial

$$x^2 + \lambda = 0.$$

Let β and $-\beta$ be two roots of $x^2 + \lambda = 0$. Thus

$$X(x) = Ce^{\beta x} + De^{-\beta x},$$

$$X'(x) = C\beta e^{\beta x} - D\beta e^{-\beta x}.$$

Therefore from the boundary conditions $X(0) = 0$ and $X'(l) = 0$, we have

$$C + D = 0, \, C\beta e^{\beta l} - D\beta e^{-\beta l} = 0.$$

Therefore

$$C = D = 0.$$

This again leads to a trivial solution $u(x, t) = 0$. □

Next we solve the exercise.

Proof. The system of linear equations:

$$\begin{cases} u_t - ku_{xx} &= 0, 0 < x < l, t > 0, \\ u(0,t) &= 0, \\ u_x(l,t) &= 0. \end{cases}$$

Let $u(x,t) = X(x)T(t)$. Then

$$X(x)T''(t) = kX''(x)T(t).$$

Thus divided by $kX(x)T(t)$ to both sides

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

From Lemma 4.1, we know that $\lambda > 0$. Let $\lambda = \beta^2$. Solving the two equations

$$\begin{cases} X''(x) &= -\beta^2 X(x), \\ T'(t) &= -\beta^2 kT(t), \end{cases}$$

we know that

$$X(x) = C\cos\beta x + D\sin\beta x,\ T(t) = Ae^{-k\beta^2 t}.$$

By using the mixed boundary conditions, $u(0,t) = 0$ implies that

$$X(0)T(t) = 0.$$

To avoid the trivial solution $u(x,t) = 0$, we have $X(0) = 0$, hence

$$C = 0.$$

This in turn implies that

$$D \neq 0.$$

By $u_x(l,t) = 0$, we have

$$X'(l)T(t) = 0.$$

This implies that $X'(l) = 0$, which yields

$$\cos\beta x = 0, \Rightarrow \beta = \frac{(n+\frac{1}{2})\pi}{l}, n \in \mathbb{Z}.$$

Therefore for $n \in \mathbb{Z}$,

$$X_n(x) = D_n \sin\frac{(n+\frac{1}{2})\pi x}{l},$$

$$T_n(t) = A_n e^{-\frac{(n+\frac{1}{2})^2\pi^2 kt}{l^2}}.$$

Hence

$$u(x,t) = \sum_{n\in\mathbb{Z}} D_n A_n \sin \frac{(n+\frac{1}{2})\pi x}{l} e^{-\frac{(n+\frac{1}{2})^2\pi^2 kt}{l^2}}$$
$$= \sum_{n=0}^{\infty} (D_n A_n + D_{-(n+1)} A_{-(n+1)}) \sin \frac{(n+\frac{1}{2})\pi x}{l} e^{-\frac{(n+\frac{1}{2})^2\pi^2 kt}{l^2}}.$$

To simplify the notation, we have

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin \frac{(n+\frac{1}{2})\pi x}{l} e^{-\frac{(n+\frac{1}{2})^2\pi^2 kt}{l^2}},$$

where A_n is some constant. □

5. Section 4.2. # 2.

Proof. The system of linear equations:

$$\begin{cases} u_t - ku_{xx} &= 0, 0 < x < l, t > 0, \\ u(0,t) &= 0, \\ u_x(l,t) &= 0. \end{cases}$$

Let $u(x,t) = X(x)T(t)$. Then

$$X(x)T'(t) = kX''(x)T(t).$$

Thus divided by $kX(x)T(t)$ to both sides

$$\frac{T'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

From Lemma 4.1, we know that $\lambda > 0$. Let $\lambda = \beta^2$. Solving the two equations

$$\begin{cases} X''(x) &= -\beta^2 X(x), \\ T''(t) &= -\beta^2 c^2 kT(t), \end{cases}$$

we know that

$$X(x) = C\cos\beta x + D\sin\beta x,\ T(t) = A\cos\beta ct + B\sin\beta ct.$$

By using the mixed boundary conditions, $u_x(0,t) = 0$ implies that

$$X'(0)T(t) = 0.$$

To avoid the trivial solution $u(x,t) = 0$, we have $X'(0) = 0$, hence

$$D = 0.$$

This in turn implies that

$$C \neq 0.$$

By $u(l,t)=0$, we have

$$X(l)T(t)=0.$$

This implies that $X(l)=0$, which, without loss of generality, yields

$$\cos\beta l = 0, \Rightarrow \beta = \frac{(n+\frac{1}{2})\pi}{l}, n = 0,1,2,\cdots$$

Therefore for $n=0,1,2,\cdots$,

$$X_n(x) = C_n \cos\frac{(n+\frac{1}{2})\pi x}{l},$$

$$T_n(t) = A_n e^{-\frac{(n+\frac{1}{2})^2\pi^2 kt}{l^2}} + B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l}.$$

Hence

$$u(x,t) = \sum_{n=0}^{\infty} \left(\cos\frac{(n+\frac{1}{2})\pi x}{l} \left(A_n \cos\frac{(n+\frac{1}{2})\pi ct}{l} + B_n \sin\frac{(n+\frac{1}{2})\pi ct}{l} \right) \right)$$

where A_n, B_n and C_n are some constants. □

6. Section 4.2. # 3.

Proof. The system of linear equations:

$$\begin{cases} u_t - iku_{xx} &= 0, 0<x<l, t>0, \\ u_x(0,t) &= 0, \\ u(l,t) &= 0, \end{cases}$$

where k is a real number.

Let $u(x,t) = X(x)T(t)$. Then

$$X(x)T'(t) = ikX''(x)T(t).$$

Thus divided by $ikX(x)T(t)$ to both sides

$$\frac{T'(t)}{ikT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

From Lemma 4.1, we know that $\lambda > 0$. Let $\lambda = \beta^2$. Solving the two equations

$$\begin{cases} X''(x) &= -\beta^2 X(x), \\ T'(t) &= -i\beta^2 kT(t), \end{cases}$$

we know that

$$X(x) = C\cos\beta x + D\sin\beta x,\ T(t) = Ae^{-ik\beta^2 t}.$$

By using the mixed boundary conditions, $u_x(0,t)=0$ implies that

$$X'(0)T(t) = 0.$$

To avoid the trivial solution $u(x,t)=0$, we have $X'(0)=0$, hence
$$D=0.$$
This in turn implies that
$$C\neq 0.$$

By $u(l,t)=0$, we have
$$X(l)T(t)=0.$$
This implies that $X(l)=0$, which, without loss of generality, yields
$$\cos\beta l=0,\Rightarrow \beta=\beta_n=\frac{(n+\frac{1}{2})\pi}{l}, n=0,1,2,\cdots$$

We also have
$$T_n(t)=A_n e^{-i\beta_n^2 t}=A_n e^{-i\frac{(n+\frac{1}{2})^2\pi^2}{l^2}t}.$$

Therefore
$$u(x,t)=\sum_{n=0}^{\infty}A_n\cos\frac{(n+\frac{1}{2})\pi x}{l}e^{-i\frac{(n+\frac{1}{2})^2\pi^2}{l^2}t}.$$
where A_n is some constant. □

7. Section 4.3. # 1.

Proof. For the equation,
$$X''=-\lambda X, \lambda>0, \lambda=\beta^2.$$
Thus
$$X(x)=C\cos\beta x+D\sin\beta x.$$
The boundary condition $X(0)=0$ implies that
$$C=0.$$
On the other hand, $X'(x)=D\beta\cos\beta x$ implies that
$$D\beta\cos\beta l+aD\sin\beta l=0.$$
From the equation above,
$$\tan\beta l=-\frac{\beta}{a}.$$
We graph the two functions, $y=\tan l\beta$ and $y=-\frac{\beta}{a}$, to find that there are $\beta_1,\beta_2,\cdots,\beta_k,\cdots$ such that
$$\frac{(2k-1)\pi}{2l}\leq\beta_k\leq\frac{k\pi}{2l}.$$
For each k, the eigenfunctions are
$$X_k(x)=A_k\sin\beta_k x,$$

with the corresponding eigenfunctons are $-\beta_k^2$. □

8. Section 4.3. # 2.

Proof. **(a). We first prove one implication.** Suppose that $X''(0) = 0$. Then $X(x) = aX + b$. Thus

$$X'(x) = a,\ 0 \leq x \leq l.$$

Hence the boundary conditions,

$$\begin{cases} X'(0) - a_0 X(0) & = 0, \\ X'(l) + a_l X(l) & = 0. \end{cases}$$

Thus we have

$$\begin{cases} a - a_0 b & = 0, \\ a + a_l(al + b) & = 0. \end{cases}$$

Hence

$$(a_0 + a_l)b = -a_0 a_l bl,$$

which implies that

$$a_0 + a_l = -a_0 a_l l. \tag{4}$$

On the other hand, we suppose that the equation (4) holds:

$$a_0 + a_l = -a_0 a_l l. \tag{5}$$

To verify that the equation

$$-X'' = \lambda X$$

has the zero eigenvalue $\lambda = 0$, we check that whether $X''(x) = 0$ has a nonzero solution. To this end, it suffices to say that the coefficients of the solution $X(x) = ax + b$, a, b, are not both zero. Indeed, for $a \neq 0$ and $b \neq 0$, any function $X(x) = ax + b$ works in the sense that $X'' = 0$ and it satisfies the Robin boundary conditions at $x = 0$ and $x = l$. □

9. Section 4.3. # 3.

Proof. Let $\lambda = -\gamma^2$. Then

$$X(x) = C \cosh \gamma x + D \sinh yx.$$

From the boundary conditions,

$$\begin{cases} X'(0) - a_0 X(0) & = 0, \\ X'(l) - a_l X(l) & = 0, \end{cases}$$

we have

$$\begin{cases} D\gamma - a_0 C & = 0, \\ (C\gamma \sinh \gamma l + D\gamma \cosh \gamma l) + a_l(C \cosh \gamma l + D \sinh \gamma l) & = 0. \end{cases}$$

This implies that

$$(C\gamma + a_l D) \sinh \gamma l + (a_0 + a_l) C \cosh \gamma l = 0.$$

Multiplied by γ, we have

$$(C\gamma^2 + a_l a_0 C) \sinh \gamma l + (a_0 + a_l)\gamma C \cosh \gamma l = 0.$$

So

$$\tanh \gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.$$

The eigenfunctions are

$$X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x.$$

□

10. Section 4.3. # 4.

Proof. We graph the following two functions:

$$f(\gamma) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}$$
$$g(\gamma) = \tanh \gamma l.$$

We look at the slopes of f and g.

$$f'(0) = -\frac{a_0 + a_l}{a_0 a_l} < l,$$
$$g'(0) = l \lim_{\gamma \to 0} \frac{\tanh \gamma l}{\gamma l} = l.$$

under the condition that $a_0 < 0, a_l < 0$ and $-a_0 - a_l < a_0 a_l l$. Since $f'(0) < g'(0)$, we see that there are two intersection points in the first quadrant. This shows that there are two negative eigenvalues. □

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