

## HOMEWORK 7

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### 1. P 139. EX. 5.1.2

a). For  $j = 0, 1, \dots, n$ , we have

$$\frac{0}{n} = 0, \frac{n}{n} = 1, \text{ and } \frac{j}{n} < \frac{j+1}{n}.$$

So this is a partition of the interval  $[0, 1]$ .

b). For a  $P$  of  $[0, 1]$ ,  $P = \{x_i : 0 \leq i \leq n, x_i = 0, x_n = 1, x_0 < x_1 < \dots < x_n\}$ . The upper integral is

$$(U) \int_0^1 f(x) dx = \inf \{U(f, P) : P \text{ is a partition of } [0, 1]\}.$$

For  $k \in \mathbb{N} \cup \{0\}$ . Let  $\mathcal{P}_k$  denote the collection of partitions of  $[0, 1]$  with  $k$  points in  $(0, 1)$ . It is clear that each partition in  $\mathcal{P}_{k+1}$  is a refinement of a partition in  $\mathcal{P}_k$ . Then by Remark 5.7,

$$\inf_{P \in \mathcal{P}_{k+1}} U(f, P) \leq \inf_{P \in \mathcal{P}_k} U(f, P).$$

Let  $u_k = \inf_{P \in \mathcal{P}_k} U(f, P)$ . The sequence  $\{u_k\}_{k \geq 0}$  is decreasing. Moreover, by Remark 5.8,

$$u_k \geq L(f, \{0, 1\}), \text{ i.e., } u_k \text{ is bounded below.}$$

So by the monotone convergence theorem,  $u = \lim_{k \rightarrow \infty} u_k$  and  $u \in \mathbb{R}$ . It is not hard to show that

$$u = (U) \int_0^1 f(x) dx.$$

Indeed, by definition of the upper integer,

$$(U) \int_0^1 f(x) dx \leq u.$$

On the other hand, for any  $\varepsilon > 0$ , there exists  $P \in \mathcal{P}_k$ ,

$$U(f, P) \leq (U) \int_0^1 f(x) dx + \varepsilon.$$

Since  $U(f, P) \geq u_k$ ,  $u \leq (U) \int_0^1 f(x)dx + \varepsilon$ . This holds for any  $\varepsilon$ , so

$$u \leq (U) \int_0^1 f(x)dx.$$

Hence  $u = (U) \int_0^1 f(x)dx$ . Similarly, we can define  $l_k := \sup_{P \in \mathcal{P}_k} L(f, P)$  and show that  $\{l_k\}$  is increasing and bounded above. Let  $l = \lim_{k \rightarrow \infty} l_k$ . We then show that

$$l = (L) \int_0^1 f(x)dx.$$

Now we show that  $l = u$ , i.e.,  $f$  is integrable. By the assumption that

$$I_0 = \lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n),$$

for any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for any  $n \geq N$ ,

$$\begin{aligned} I_0 - \varepsilon &< U(f, P_n) < I_0 + \varepsilon, \\ I_0 - \varepsilon &< L(f, P_n) < I_0 + \varepsilon. \end{aligned}$$

So

$$(1) \quad U(f, P_n) \leq L(f, P_n) + 2\varepsilon.$$

On the other hand,  $U(f, P_n) \geq u_n \geq u$  and  $L(f, P_n) \leq l_n \leq l$ , so

$$u \leq l + 2\varepsilon.$$

So it implies that  $u \leq l$ . It is clear that  $l \leq u$ . So  $l = u$ . So  $f$  is integrable.

Another proof is short without using the definition  $l$  and  $u$ . By (1),

$$U(f, P_n) \leq L(f, P_n) + 2\varepsilon$$

By the definitions of lower and upper integrals,

$$(U) \int_0^1 f(x)dx \leq (L) \int_0^1 f(x)dx + 2\varepsilon, \text{ for any } \varepsilon > 0.$$

So

$$(U) \int_0^1 f(x)dx = (L) \int_0^1 f(x)dx.$$

c).

$\alpha$ ). For  $f(x) = x$ , For the partition  $P_n$ ,

$$M_j(f)\Delta x_j = \frac{j}{n} \times \frac{1}{n} = \frac{j}{n^2}; m_j(f)\Delta x_j = \frac{j-1}{n^2}.$$

So

$$U(f, P_n) = \sum_{j=1}^n M_j \Delta x_j = \frac{\sum_{j=1}^n j}{n^2} = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}.$$

Similarly

$$L(f, P_n) = \frac{n-1}{2n}.$$

So

$$\int_0^1 f(x) dx = \frac{1}{2}.$$

$\beta$ ). For  $f(x) = x^2$ ,

$$M_j(f) \Delta x_j = \frac{j^2}{n^2} \times \frac{1}{n} = \frac{j^2}{n^3}; m_j(f) \Delta x_j = \frac{(j-1)^2}{n^3}.$$

So

$$U(f, P_n) = \sum_{j=1}^n M_j \Delta x_j = \frac{\sum_{j=1}^n j^2}{n^3} = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}.$$

Similarly

$$L(f, P_n) = \frac{(n-1)n(2n-1)}{6n^3}.$$

So

$$\int_0^1 f(x) dx = \frac{1}{3}.$$

$\gamma$ ). For the function  $f$ , let  $x_n$  be the point in  $P_n$  such that  $x_n$  is the first number satisfying  $x_n \geq \frac{1}{2}$ . Then

$$L(f, P_n) = 0 \times (x_n - 1/2) + (1 - x_n) = 1 - x_n.$$

and

$$U(f, P_n) = 1 \times (x_n - 1/2) + 1 - x_n = \frac{1}{2}.$$

For  $n = 2k$ ,  $x_n = \frac{k}{2k} = \frac{1}{2}$ . For  $n = 2k - 1$ ,  $x_n = \frac{k}{2k-1}$ . In any case,  $\lim_{n \rightarrow \infty} x_n = \frac{1}{2}$ . So

$$\int_0^1 f(x) dx = \frac{1}{2}.$$

## 2. P139. Ex. 5.1.3

*Proof.* It is obvious that  $f$  is bounded. By 5.1.2(b), we need to show that

$$\lim_{n \rightarrow \infty} L(f, P_n) = \lim_{n \rightarrow \infty} U(f, P_n).$$

Firstly,

$$L(f, P_n) = 0, \text{ for all } n. \text{ So } \lim_{n \rightarrow \infty} L(f, P_n) = 0.$$

For for the upper integer ( $U$ )  $\int_0^1 f(x) dx$ : for each  $n$ , we divide  $[0, 1]$  into  $n$  sub-intervals  $\{I_j = [\frac{j-1}{n}, \frac{j}{n}]\}_{1 \leq j \leq n}$ . Now for any  $N \in \mathbb{N}$ , if  $n \geq N$ ,

$$\frac{1}{n} \leq \frac{1}{N}.$$

That is to say, the points  $\{\frac{1}{N}, \frac{1}{N+1}, \dots\}$  are in the interval  $[0, \frac{1}{N}]$ . The other  $N - 1$  points  $\{1, \frac{1}{2}, \dots, \frac{1}{N-1}\}$  are in the interval  $[\frac{1}{N}, 1]$ . For any  $n \geq N$ ,

$$(U) \int_0^1 f(x)dx \leq U(f, P_n) = \sum_{j=1}^{N-1} \frac{M_j}{n} + \sum_{j=N}^n \frac{M_j}{n} \leq \frac{N-1}{n} + \frac{1}{N},$$

since  $M_j \leq 1$  for each  $1 \leq j \leq n$ . Let  $n \rightarrow \infty$ ,

$$(U) \int_0^1 f(x)dx \leq U(f, P_n) \leq \frac{1}{N}.$$

Since  $N \in \mathbb{N}$  is arbitrary,

$$(U) \int_0^1 f(x)dx \leq U(f, P_n) = 0.$$

This proves that  $f$  is integrable, and  $\int_0^1 f(x)dx = 0$ .

Note that this proof makes use of Ex. 5.1.2. Another way is the following. We show that, for any  $\varepsilon > 0$ , there exists a partition  $P$  of  $[0, 1]$  such that,

$$U(f, P) - L(f, P) < \varepsilon.$$

For  $\varepsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that  $\frac{1}{N} < \varepsilon/2$ . So for any  $n \geq N$ ,

$$\frac{1}{n} < \varepsilon/2.$$

Next we choose sufficiently small and disjoint closed intervals  $\{I_j\}_{j=1}^{N-1}$  centered at  $\frac{1}{j}$ ,  $1 \leq j \leq N - 1$ , such that

$$\sum_{j=1}^{N-1} |I_j| < \frac{\varepsilon}{2}.$$

Let  $P$  be the partition contains the two points  $0, \frac{1}{N}$  and the endpoints of  $\{I_j\}_{j=1}^{N-1}$ . For  $P$ ,

$$U(f, P) - L(f, P) = 1 \times \frac{1}{N} + \sum_{j=1}^{N-1} M_j(f)|I_j| + 0 = \frac{1}{N} + \sum_{j=1}^{N-1} |I_j| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

On the other hand, for  $P$ ,

$$L(f, P) = 0.$$

So

$$U(f, P) - L(f, P) < \varepsilon.$$

So  $f$  is integrable. To compute  $\int_0^1 f(x)dx$ ,

$$\int_0^1 f(x)dx = (U) \int_0^1 f(x)dx \leq U(f, P) < \varepsilon.$$

That is to say,

$$\int_0^1 f(x)dx < \varepsilon.$$

This holds for any  $\varepsilon > 0$ , which, together with  $\int_0^1 f(x)dx \geq 0$ , shows that

$$\int_0^1 f(x)dx = 0.$$

□

### 3. P139. Ex. 5.1.4

*Proof. a).* The function  $f$  is continuous at  $x_0 \in [a, b]$  and  $f(x_0) \neq 0$ ; so  $|f|$  is continuous at  $x_0$  and  $|f(x_0)| > 0$ , say,  $|f(x_0)| \geq \varepsilon > 0$  for some  $\varepsilon > 0$ . By continuity of  $|f|$  at  $x_0$ , there exists  $\delta > 0$  such that

$$|f(x)| > \varepsilon/2, \text{ for } x \in [x_0, x_0 + \delta], \text{ or } [x_0, x_0 - \delta].$$

Denote the interval by  $I_\delta$ . and choose a partition  $P$  of  $[a, b]$ :  $I_\delta$ , and the closed intervals  $I$  coming from  $[a, b] \setminus I_\delta$ . With respect to this partition,

$$L(|f|, P) = \inf_{x \in I} |f(x)||I| + \inf_{x \in I_\delta} |f(x)||I_\delta| \geq 0 + \frac{\varepsilon\delta}{2}.$$

So

$$(L) \int_0^1 f(x)dx \geq \frac{\varepsilon\delta}{2} > 0.$$

**b).** If  $f = 0$  on  $[a, b]$ ,

$$\int_a^b f(x)dx = 0.$$

The converse direction follows from part (a).

**c).** If without absolute value, part (b) fails. For instance, choose  $f(x) = x$  on  $[-1, 1]$ . □

### 4. P139 Ex. 5.15.

*Proof.* If  $f(x) = 0$  for all  $x \in [a, b]$ ,

$$\int_a^c f(x)dx = 0, \text{ for all } c \in [a, b].$$

Conversely, the proof is similar to the part (a) in the previous exercise. If  $f(x_0) > 0$  for some  $x_0 \in [a, b]$ . By the continuity of  $f$  at  $x_0$ ,

$$f(x) > \varepsilon/2, \text{ for } x \in I_\delta.$$

Let  $I_\delta = [x_0, x_0 + \delta]$ . Then

$$0 = \int_a^{x_0+\delta} f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^{x_0+\delta} f(x)dx = 0 + \int_{x_0}^{x_0+\delta} f(x)dx \geq \frac{\delta\varepsilon}{2} > 0.$$

This is a contradiction. So  $f(x) = 0$  for all  $x \in [a, b]$ .  $\square$

## 5. P139 EX. 5.16

*Proof.* Let  $E := \{x_1, \dots, x_N\}$ .  $E \subset [a, b]$ .  $f$  is integrable on  $[a, b]$ . For any  $\varepsilon > 0$ , there exists a partition  $P$  such that

$$U(f, P) - L(f, P) \leq \varepsilon/2.$$

For each  $x_i$ , we choose an interval  $[x_i, x_i + \frac{1}{n}]$  or  $[x_i - \frac{1}{n}, x_i]$ , where  $n$  is large such that these intervals are contained in  $[a, b]$ . Moreover, if setting the bound of  $g$  to be  $M$ , then we choose  $n$  large enough that

$$\sum_{j=1}^N \frac{2M}{n} \leq \varepsilon/2.$$

We add these endpoints of these intervals to the partition  $P$ , then it is a refinement of  $P$ , which we denote by  $Q$ . We have

$$U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P) \leq \varepsilon/2.$$

On the other hand, since  $g = f$  on  $[a, b] \setminus E$  and  $Q$  is a partition of  $[a, b]$ .

$$U(g, Q) - L(g, Q) \leq U(f, Q) - L(f, Q) + \sum_{j=1}^N \frac{2M}{n} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So  $g$  is integrable on  $[a, b]$ .

Let the bound of  $f$  be  $M_1$ . For any  $\varepsilon > 0$ , there exists a partition  $P$  such that

$$(2) \quad \int_a^b f(x)dx \leq U(f, P) \leq \int_a^b f(x)dx + \varepsilon, \text{ and } \int_a^b g(x)dx \leq U(g, P) \leq \int_a^b g(x)dx + \varepsilon.$$

On the other hand, from the previous reasoning, we choose intervals and  $n$  large enough such that

$$\sum_{j=1}^N \frac{M + M_1}{n} \leq \varepsilon.$$

After refining the partition  $P$  by  $Q$ , we have

$$|U(f, Q) - U(g, Q)| \leq \varepsilon.$$

The equation (2) of course holds when  $P$  is replaced by  $Q$ . So we have

$$\begin{aligned}\int_a^b f(x)dx &\leq U(f, Q) \leq U(g, Q) + \varepsilon \leq \int_a^b g(x)dx + 2\varepsilon. \\ \int_a^b g(x)dx - \varepsilon &\leq U(g, Q) - \varepsilon \leq U(f, Q) \leq \int_a^b f(x)dx + \varepsilon.\end{aligned}$$

This implies that  $\int_a^b f(x)dx = \int_a^b g(x)dx$ . □

### 6. P140. Ex. 5.1.7

*Proof. a).* The key point in part (a) is the following two inequalities, for any interval  $I$  in  $[a, b]$ ,

$$\begin{aligned}\sup_{x \in I} (f(x) + g(x)) &\leq \sup_{x \in I} f(x) + \sup_{x \in I} g(x). \\ \inf_{x \in I} (f(x) + g(x)) &\geq \inf_{x \in I} f(x) + \inf_{x \in I} g(x).\end{aligned}$$

We omit the proofs here.

**b).** We prove the first equation here. The proof for the second equation is similar. For any  $\varepsilon > 0$ , there exists a partition  $P$  such that

$$U(f, P) \leq (U) \int_a^b f(x)dx + \varepsilon.$$

Considering  $Q = P \cup \{c\}$  and splitting  $Q = Q_1 \cup Q_2$ , where  $Q_1$  is a partition of  $[a, c]$  and  $Q_2$  is a partition of  $[c, b]$ . We have

$$U(f, Q) = U(f, Q_1) + U(f, Q_2) \geq (U) \int_a^c f(x)dx + (U) \int_c^b f(x)dx.$$

So

$$\int_a^b f(x)dx + \varepsilon \geq (U) \int_a^c f(x)dx + (U) \int_c^b f(x)dx,$$

holds for any  $\varepsilon > 0$ . This proves that

$$\int_a^b f(x)dx \geq (U) \int_a^c f(x)dx + (U) \int_c^b f(x)dx$$

The reverse inequality is established similarly. □

7. P141. Ex. 5.1.9

*Proof.* The function  $f$  is integrable on  $[a, b]$ : for any  $\varepsilon > 0$ , there exists a partition  $P$  such that

$$U(f, P) - L(f, P) \leq 2\sqrt{c}\varepsilon.$$

On each sub-interval  $I$  from the partition  $P$ ,

$$\begin{aligned} \sup_{x \in I} \sqrt{f(x)} - \inf_{x \in I} \sqrt{f(x)} &= \sqrt{\sup_{x \in I} f(x)} - \sqrt{\inf_{x \in I} f(x)} \\ &= \frac{\sup_{x \in I} f(x) - \inf_{x \in I} f(x)}{\sqrt{\sup_{x \in I} f(x)} + \sqrt{\inf_{x \in I} f(x)}} \\ &\leq \frac{\sup_{x \in I} f(x) - \inf_{x \in I} f(x)}{2\sqrt{c}}. \end{aligned}$$

Then with respect to the same partition  $P$ ,

$$U(\sqrt{f}, P) - L(\sqrt{f}, P) \leq \varepsilon.$$

So the function  $\sqrt{f}$  is Riemann integrable.  $\square$

8. P150. Ex. 5.2.2

*Proof. a).* We prove it by contradiction. By continuity of  $f$  and the intermediate value theorem, we may assume that  $f$  is either strictly positive, or strictly negative. Suppose that we are in the first situation. By the extreme value theorem, there exists  $x_0$  such that

$$f(x_0) = \inf_{x \in [a, b]} f(x) > 0.$$

Then

$$\int_a^b f(x)x^n dx \geq \int_a^b f(x_0)x^n dx \geq f(x_0) \int_a^b f(x) dx = \frac{f(x_0)(b^{n+1} - a^{n+1})}{n+1} > 0.$$

This is because  $f(x_0) > 0$  and the function  $g(x) = x^{n+1}$  with  $n$  being even is a strictly increasing function on  $\mathbb{R}$ . Hence we have a contradiction. So there is a point  $x \in [a, b]$  such that  $f(x) = 0$ .

**b).** Take  $f(x) = 1$  and  $n = 1$ , and  $[a, b] = [-1, 1]$ .

**c).** The same proof as the part (a).  $\square$

9. P151. Ex. 5.2.4

*Proof.* Fix  $x \in [0, \infty)$ . Define the following function  $\phi : [0, x] \rightarrow \mathbb{R}$ ,

$$\phi(t) := \int_0^x e^{-y^2} f(y) dy - \int_t^x f(y) dy.$$

By Theorem 5.26,  $\phi$  is continuous on  $[a, b]$ . Also

$$\phi(x) = \int_0^x e^{-y^2} f(y) dy \geq 0.$$

Since  $f(x) \geq 0$  for all  $x$ , and  $e^{-x^2} \leq 1$ ,

$$\phi(0) = \int_0^x e^{-y^2} f(y) dy - \int_0^x f(y) dy \leq 0.$$

So by the intermediate value theorem, there exists  $t \in [0, x]$ , which depends on  $x$  such that

$$\phi(t) = 0.$$

That is to say,

$$\int_0^x e^{-y^2} f(y) dy = \int_t^x f(y) dy.$$

Let  $E(x)$  be the set of such  $t$ ; it is clear that  $E(x)$  is bounded below by 0. So  $\inf E(x)$  exists, which is also unique. Define the function  $g$  mapping  $x$  to  $g(x) = \inf E(x)$ . By the continuity of the integral  $\int_t^x f(y) dy$ , we see that

$$\int_0^x e^{-y^2} f(y) dy = \int_{g(x)}^x f(y) dy.$$

□

10. P151. Ex. 5.2.5

*Proof.* The function  $f$  is integrable on  $[0, 1]$ , then  $f$  is bounded, say, by  $M$ . Then

$$n^\alpha \int_0^{1/n^\beta} f(x) dx \leq n^\alpha \frac{M}{n^\beta} = Mn^{\alpha-\beta}$$

which goes to zero as  $\alpha < \beta$ . □

11. P151. Ex. 5.2.6

*Proof.* **a).** The integrability of  $f$  on  $[a, b]$  implies that  $f$  is bounded by  $M$ . Because  $g_n \geq 0$ ,

$$|f(x)g_n| \leq Mg_n.$$

and  $fg_n$  is also integrable on  $[a, b]$  by Corollary 5.23. We have

$$\int_a^b |f(x)g_n(x)|dx \leq M \int_a^b g_n(x)dx$$

which goes to zero by hypothesis.

**b).** Let  $g_n(x) = x^n$ , then

$$\int_0^1 g_n(x)dx = \frac{1}{n+1}$$

which goes to zero as  $n$  goes to infinity. Hence we can apply the result in part (a).  $\square$

## 12. P151. Ex. 5.2.7

*Proof.* By Theorem 5.20,

$$\sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} f(x)dx = \int_a^{x_n} f(x)dx.$$

Since  $F(x) = \int_a^x f(t)dt$  is continuous in  $x$ , in particular it is continuous at  $x = b$ . So if fixing  $b$  and  $x_n \rightarrow b$ , by sequential characterization of limits, we have

$$\lim_{n \rightarrow \infty} \int_a^{x_n} f(t)dt = \int_a^b f(t)dt.$$

$\square$

## 13. P151. Ex. 5.2.8

*Proof.* The function  $f$  is continuous on  $[a, b]$ , so is  $|f|$ . By the extreme value theorem,

$$M = \sup_{x \in [a, b]} |f(x)| = |f(x_0)|$$

for some  $x_0 \in [a, b]$ . If  $M > 0$ , by the continuity of  $|f|$ , for any  $\varepsilon > 0$ , there exists a nondegenerate interval containing  $x_0$  in  $[a, b]$ , which we denote by  $I$ , such that, for any  $x \in I$ ,

$$M - \varepsilon \leq |f(x)| \leq M.$$

From  $\int_a^b |f(x)|^p dx = \int_I |f(x)|^p dx + \int_{[a, b] \setminus I} |f(x)|^p dx \geq \int_I |f(x)|^p dx$ ,

$$\int_a^b |f(x)|^p dx \geq (M - \varepsilon)^p |I|.$$

On the other hand, it is clear that

$$\int_a^b |f(x)|^p dx \leq M^p(b-a).$$

So the proof is complete for part (a).

**b).** From part (a), we have

$$(M - \varepsilon)|I|^{1/p} \leq \left( \int_a^b |f(x)|^p dx \right)^{1/p} \leq M(b-a)^{1/p}.$$

On the other hand, for any  $x > 0$ , Example 2.21 implies that for  $n \in \mathbb{N}$ , we have  $\lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1$ . Given any large  $p > 0$  there exists  $n \in \mathbb{N}$  such that  $n \leq p < n+1$ . So  $x^{1/p}$  is between  $x^{1/n}$  and  $x^{1/(n+1)}$ . Thus

$$\lim_{p \rightarrow \infty} x^{1/p} = 1.$$

So let  $p \rightarrow \infty$ ,

$$(M - \varepsilon) \leq \lim_{p \rightarrow \infty} \left( \int_a^b |f(x)|^p dx \right)^{1/p} \leq M.$$

This holds for any  $\varepsilon > 0$ . Hence the claim in part (b) follows.  $\square$

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