

## HOMEWORK 8

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### 1. P 161. Ex. 5.3.3

*Proof. a).* Let  $u = x^2 + 1$ .  $du = 2xdx$ .

$$\int_0^1 x^3 f(x^2 + 1) dx = \int_1^2 (u - 1) f(u) \frac{du}{2} = \frac{1}{2} \int_1^2 u f(u) du - \frac{1}{2} \int_1^2 f(u) du.$$

By hypothesis,

$$= \frac{1}{2} \times (3 - 1) - \frac{3}{2} = -\frac{1}{2}.$$

**b).** Let  $u = \sqrt{1 - x^2}$ . Then

$$du = \frac{-x}{\sqrt{1 - x^2}}, \text{ and } x^2 = 1 - u^2.$$

So

$$\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1 - x^2}} f(\sqrt{1 - x^2}) dx = \int_{1/2}^1 (1 - u^2) f(u) du.$$

By hypothesis,

$$= 3 - 7 = -4.$$

□

### 2. P161. Ex. 5.3.4

*Proof. a).* By integration by parts,

$$\int_1^e f'(x) \log x dx = f(x) \log x \Big|_1^e - \int_1^e f(x) \frac{dx}{x} = f(e) - \int_1^e f(x) \frac{dx}{x} > 0.$$

**b).** By integration by parts,

$$\int_1^1 e^x (f(x) + f'(x)) dx = \int_0^1 e^x f(x) dx + \int_0^1 e^x f'(x) dx = \int_0^1 e^x f(x) dx + e^x f(x) \Big|_0^1 - \int_0^1 e^x f(x) dx = 0.$$

**c).** This follows from the integration by parts directly.

□

## 3. P162. Ex. 5.3.5

*Proof.* In order to apply the first mean value theorem for integrals, we set  $g(x) = 1$ .

$$f(b) - f(a) = \int_a^b f'(t) dt = \int_a^b f'(t)g(t) dt = f'(x_0) \int_a^b g(t) dt = f'(x_0)(b-a).$$

□

## 4. P162. Ex. 5.3.6

*Proof.* Let  $g(t) = \alpha \int_a^t f(x) dx + \beta \int_t^b f(x) dx$  for  $c \in (a, b)$ . Then  $g(t)$  is continuous on  $(a, b)$  and

$$g'(t) = \alpha f(t) - \beta f(t) = (\alpha - \beta)f(t).$$

On the other hand,  $g(t) = 0$  for all  $t \in (a, b)$ . So the condition  $\alpha \neq \beta$  implies that

$$f(t) = 0, \text{ for } t \in (a, b).$$

Hence by the continuity of  $f$  on  $[a, b]$ ,  $f(a) = 0$  and  $f(b) = 0$ . Therefore  $f(x) = 0$  for all  $x \in [a, b]$ . □

## 5. P163. Ex. 5.3.9

*Proof.* The function  $f$  is continuously differentiable on  $[a, b]$ , and  $1 - 1$  on  $[a, b]$ . Then  $f^{-1}$  exists and  $f^{-1}$  is continuously differentiable. By Theorem 5.34,

$$\int_{f(a)}^{f(b)} f^{-1}(x) dx = \int_a^b f^{-1}(f(y)) f'(y) dy = \int_a^b y f'(y) dy.$$

Then the equation follows from the integration by parts. □

## 6. P163. Ex. 5.3.10

*Proof.* By Theorem 5.3.4,  $f \circ \phi \cdot |\phi'|$  is integrable on  $[a, b]$ . On the other hand,  $\phi'$  is continuous on  $[a, b]$  and  $\phi'$  is never zero on  $[a, b]$ ; so we may assume

that  $\phi'$  is strictly positive on  $[a, b]$  by the intermediate value theorem. It also holds that  $\frac{1}{\phi'}$  is continuous and hence integrable on  $[a, b]$ . So

$$f \circ \phi = (f \circ \phi \cdot |\phi'|) \times \left( \frac{1}{\phi'} \right)$$

is integrable on  $[a, b]$ . □

### 7. P168. Ex. 5.4.1

*Proof. a).*

$$\int_1^\infty \frac{1+x}{x^3} dx = \int_1^\infty x^{-3} dx + \int_1^\infty x^{-2} dx = -\frac{x^{-2}}{2} \Big|_1^\infty - \frac{1}{x} \Big|_1^\infty = \frac{1}{2} + 1 = \frac{3}{2}.$$

**d).**

$$\int_0^1 \log x dx = \int_0^1 x' \log x dx = x \log x \Big|_0^1 - \int_0^1 dx = -1$$

since  $\lim_{x \rightarrow 0^+} x \log x = 0$  by using the L'Hopital's rule. □

### 8. P169. Ex. 5.4.6

*Proof. a).* Since  $L = \lim_{x \rightarrow b^-} \frac{f(x)}{g(x)}$  and  $L \in [0, \infty)$ , for  $\varepsilon = \frac{L}{2}$ , there exists  $b - a > \delta > 0$  such that

$$\frac{f(x)}{g(x)} \leq \frac{3L}{2}, \text{ for } b - \delta < x < b.$$

So on the interval  $(b - \delta, b)$ , by Theorem 5.43,  $f$  is improperly integrable since  $g$  is improperly integrable. On the interval  $[a, b - \delta]$ ,  $f$  is integrable since  $f$  is improperly integrable on  $[a, b]$ . So  $f$  is improperly integrable on  $[a, b]$ .

**b).** If  $0 < L \leq \infty$ , there exists a positive number  $\alpha > 0$  such that

$$L > 2\alpha.$$

So  $L = \lim_{x \rightarrow b^-} \frac{f(x)}{g(x)}$  implies that there exists  $b - a > \delta > 0$  such that

$$\frac{f(x)}{g(x)} > \alpha$$

for  $x \in (b - \delta, b)$ . That is to say,  $f(x) > \alpha g(x)$  for  $x \in (b - \delta, b)$ . So if  $f$  is improperly integrable on  $[a, b]$ , Theorem 5.43 implies that  $g$  is improperly integrable on  $[a, b]$ . □

9. P169. Ex. 5.4.7

*Proof. a).* Suppose that  $L > 0$ . The limit  $\lim_{x \rightarrow \infty} f(x) = L$  implies that, there exists  $N > 0$  such that

$$f(x) > L/2$$

for all  $x > N$ . Since  $f$  is improperly integrable on  $[0, \infty)$ , Theorem 5.43 implies that the function  $g(x) = L/2$  is also improperly integrable on  $[N, \infty)$ , i.e.,

$$\int_N^\infty g(x) dx < \infty.$$

This is a contradiction.

**b).** Firstly  $f$  is locally integrable on  $[0, \infty)$ . Secondly, for any  $d > 100$ ,

$$\int_0^d f(x) dx = \sum_0^n \int_0^d f(x) dx = \sum_{N: N+2^{-N} \leq d} \int_0^d f(x) dx = \sum_{k=1}^N \frac{1}{2^k} \leq 1.$$

So  $\lim_{d \rightarrow \infty} \int_0^d f(x) dx = \sum_{k=1}^\infty \frac{1}{2^k} = 1$ . So  $f$  is improperly integrable on  $[0, \infty)$ . But  $\lim_{x \rightarrow \infty} f(x)$  does not exist because there exists two sequences  $\{x_n\}$  and  $\{y_n\}$  both going to  $\infty$  but  $f(x_n) \equiv 1$  and  $f(y_n) \equiv 0$ .  $\square$

10. P170. Ex. 5.48

*Proof.* Let  $t = x^n$ , then  $dt = nx^{n-1} dx$ ;  $dx = \frac{dt}{nt^{\frac{n-1}{n}}}$ , and

$$\int_1^\infty f(x^n) dx = \frac{1}{n} \int_1^\infty f(t) \frac{dt}{t^{(n-1)/n}}.$$

Since  $t \geq 1$ ,

$$\frac{1}{n} \int_1^\infty f(t) \frac{dt}{t^{(n-1)/n}} \leq \frac{1}{n} \int_1^\infty |f(t)| \frac{dt}{t^{(n-1)/n}} \leq \frac{1}{n} \int_1^\infty |f(t)| dt.$$

Since  $f$  is absolutely integrable on  $[1, \infty)$ ,

$$\int_1^\infty |f(t)| dt < \infty.$$

So

$$\lim_{n \rightarrow \infty} \int_1^\infty f(x^n) dx = 0.$$

$\square$

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