HOMEWORK 9

SHUANGLIN SHAO

1. P190. Ex. 6.1.4

Proof. Let $b_k = a_{k+1} - 2a_k + a_{k-1}$, where $k \in \mathbb{N}$. Then

$$s_n = \sum_{k=1}^{n} b_k = (a_2 - 2a_1 + a_0) + (a_3 - 2a_2 + a_1) +$$
$$+ (a_4 - 2a_3 + a_2) + \cdots + (a_{n+1} - 2a_n + a_{n-1})$$
$$= a_0 - a_1 + a_{n+1} - a_n \to a_0 - a_1$$

as $n \to \infty$. So for all n, $\lim_{n \to \infty} s_n$ exists and equals $a_0 - a_1$. So the series $\sum_{k=1}^{\infty} b_k$ converges and the value is $a_0 - a_1$. $\square$

2. P190. Ex. 6.1.5

Proof.
$$(x^k - x^{k+1})(x^k + x^{k+1}) = x^{2k} - x^{2k+2}.$$
If $|x| < 1$, $\sum_{k=1}^{\infty} x^{2k}$ and $\sum_{k=1}^{\infty} x^{2k+2}$ both converge. So

$$\sum_{k=1}^{\infty} (x^2 k - x^{2k+2}) \text{ converges.}$$

If $|x| > 1$, for all $k \in \mathbb{N}$,

$$|x^{2k} - x^{2k+2}| = |x^{2k}(x^2 - 1)| > |x^2 - 1| > 0.$$

So the limit $x^{2k} - x^{2k+2}$ does not exist. Hence the series diverges. If $x = \pm 1$, $x^{2k} - x^{2k+2} = 0$. So the series $\sum_{k=1}^{\infty} 0$ converges. $\square$

3. P190. Ex. 6.1.7

Proof. **a).** The function f is a differentiable function; so F is differentiable. By the mean value theorem, For x, y in I,

$$F(x) - F(y) = F'(c)(x - y).$$

Here $F'(c) = 1 - \frac{f'(c)}{f'(a)}$. We know that for all $x \in I$, $\frac{f'(x)}{f'(a)} \in [1 - r, 1]$, hence

$$0 \le 1 - \frac{f'(c)}{f'(a)} \le r.$$

This implies that

$$|F(x) - F(y)| = |F'(c)||x - y| \le r|x - y|.$$

b). This follows from part a) as

$$x_{n+1} - x_n = F(x_n) - F(x_{n-1}).$$

c). We know that $x_n = F(x_{n-1})$. By $F(I) \subset I$, if $x_{n-1} \in I$, $x_n \in I$. Since $x_0 \in I$, we see that $x_n \in I$ for all n. By Bolzano-Weierstrass theorem, there is a subsequence of $\{x_n\}_n$ which converges to a fixed number b in $I \subset \mathbb{R}$. On the other hand, by part b), $\{x_n\}$ is Cauchy and so it converges. It will converges to the same number as the subsequence does. So

$$\lim_{n \to \infty} x_n = b.$$

Taking limits on both sides of $x_n = F(x_{n-1})$ and using that F is continuous on I, so

$$b = F(b),$$

i.e.,

$$b = b - \frac{f(b)}{f'(a)},$$

which implies that $f(b) = 0$. $\qquad\square$

4. P191. Ex. 6.1.9

Proof. **a).** For $n > N$,

$$nb - \sum_{k=1}^{N} b_k = \sum_{k=1}^{N}(b - b_k) + \sum_{k=N+1}^{n}(b - b_k).$$

Since $|b_k - b| \le M$ for $k > N$, by the triangle inequality,

$$\left| nb - \sum_{k=1}^{N} b_k \right| \le \sum_{k=1}^{N} |b_k - b| + M(n - N).$$

b). The limit $\lim_{n \to \infty} b_n = b$: for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for any $n \ge N$,

$$|b_n - b| \le \varepsilon.$$

2

So if $n > N$, by part a),

$$\frac{b_1 + b_2 + \cdots + b_n - nb}{n} \leq \frac{\sum_{k=1}^{N} |b_k - b|}{n} + \varepsilon(1 - \frac{N}{n}).$$

If $n \to \infty$,

$$\limsup_{n \to \infty} \frac{b_1 + b_2 + \cdots + b_n}{n} - b \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{n \to \infty} \frac{b_1 + b_2 + \cdots + b_n}{n} - b = 0.$$

Hence

$$\lim_{n \to \infty} \frac{b_1 + \cdots b_n}{n} = b.$$

c). The converse is false. Example, $b_k = (-1)^{k-1}$. So

$$\lim_{n \to \infty} \frac{b_1 + \cdots b_n}{n} = 0.$$

but the limit of $\{b_k\}$ does not exist. $\qquad \square$

5. P191. Ex. 6.1.11

Proof. We may assume that $a_k \geq 0$ for $k \geq N$. By the limit theorems,

$$\lim_{j \to \infty} \sum_{k=1}^{N} \frac{a_k}{j+k} = 0.$$

So without loss of generality, we assume that $a_k \geq 0$ for all $k \geq 1$.

Since the series $\sum_{k=1}^{\infty} \frac{a_k}{k}$ converges: for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\sum_{k=K+1}^{\infty} \frac{a_k}{k} < \varepsilon.$$

Since for any j, $\frac{a_k/k}{j/k+1} \leq \frac{a_k}{k}$,

$$\sum_{k=K+1}^{\infty} \frac{a_k/k}{j/k+1} < \varepsilon.$$

Write

$$\sum_{k=1}^{\infty} \frac{a_k}{j+k} = \sum_{k=1}^{\infty} \frac{a_k/k}{j/k+1} = \sum_{k=1}^{K} \frac{a_k/k}{j/k+1} + \sum_{k=K+1}^{\infty} \frac{a_k/k}{j/k+1}.$$

3

We see that

$$0 \leq \sum_{k=1}^{\infty} \frac{a_k}{j+k} \leq \sum_{k=1}^{K} \frac{a_k/k}{j/k+1} + \varepsilon.$$

If $j \to \infty$, by the limit theorems,

$$0 \leq \limsup_{j\to\infty} \sum_{k=1}^{\infty} \frac{a_k}{j+k} \leq \varepsilon.$$

This holds for any $\varepsilon \geq 0$. So

$$\lim_{j\to\infty} \sum_{k=1}^{\infty} \frac{a_k}{j+k} = 0.$$

$\square$

6. P192. Ex. 6.1.12

Proof. Let $b_n = \sum_{k=1}^{n} ka_k = \frac{n+1}{n+2}$, then

$$b_{n+1}-b_n = ((n+1)a_{n+1} + na_n + \cdots + 2a_2 + a_1)-(na_n + \cdots + 2a_2 + a_1) = (n+1)a_{n+1}.$$

This implies that

$$a_{n+1} = \frac{1}{n+1}\left(\frac{n+2}{n+3} - \frac{n+1}{n+2}\right) = \frac{1}{(n+1)}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+3)}.$$

So

$$a_{n+1} = \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right).$$

This holds for $n \geq 1$. So

$$a_n = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) \text{ and } a_1 = \frac{2}{3}.$$

So

$$\sum_{k=2}^{n} a_k = a_2 + \cdots + a_n$$

$$= \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{2}(\frac{1}{2} - \frac{1}{4})\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}(\frac{1}{n} - \frac{1}{n+2})\right)$$

$$= \frac{1}{2} - \frac{1}{n+1} - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2}\right).$$

So if $n \to \infty$,

$$\sum_{n=2}^{\infty} a_k = \frac{1}{2} - \frac{5}{12} = \frac{1}{12}.$$

4

So
$$\sum_{k=1}^{\infty} a_k = \frac{2}{3} + \sum_{k=2}^{\infty} a_k = \frac{3}{4}.$$

☐

7. P197. Ex. 6.2.1

Proof. **a).** For large k,
$$\frac{2k+5}{3k^3 + 2k - 1} \leq \frac{c}{k^2}$$
for some $c > 0$. So this series converges.

d). Since $e^k \geq ck^6$ for some $c > 0$,
$$\frac{k^3 \log^2 k}{e^k} \leq \frac{c}{k^2}.$$

So this series converges. ☐

8. P197. Ex. 6.2.2

Proof. **a).** For large k,
$$\frac{3k^3 + k - 4}{5k^4 - k^2 + 1} \geq \frac{c}{k}$$
for some $c > 0$. So this series diverges.

d). Since the integral $\int_2^{\infty} \frac{1}{x \log^p x} dx = \int_{\log 2}^{\infty} \frac{1}{x^p} dx = \infty$, the series diverges.
☐

9. P197. Ex. 6.2.3

Proof. We assume that $a_k \leq M$ for some $M > 0$. Since
$$\frac{a_k}{(k+1)^p} \leq \frac{M}{k^p}.$$

So this series converges. ☐

10. P198. Ex. 6.2.5

Proof. For $p \geq 0$,
$$\frac{|a_k|}{k^p} \leq |a_k|.$$
Since $\sum |a_k|$ converges, the series converges.

If $p < 0$, the series may converge or diverge. For $p = -2$, $a_k = \frac{1}{k^2}$, $\sum a_k$ converges but $a_k/k^p = 1$. So the series diverges. For this same p, if $a_k = \frac{1}{k^4}$, the series is $\sum \frac{1}{k^2}$. So the series converges. $\square$

11. P198. Ex. 6.2.7

Proof. The series $\sum a_k$ converges, so a_k is bounded by M for all k. Since $0 \leq a_k b_k \leq M b_k$ and $\sum b_k$ converges, $\sum a_k b_k$ converges. $\square$

12. P198. Ex. 6.2.9

Proof. If $\sum a_k$ converges, the partial sum $S_n = \sum_{k=1}^{n} a_k$ converges. Since the partial sum of the second series is
$$(a_2 + a_3) + (a_4 + a_5) + \cdots + (a_{2k} + a_{2k+1}) = S_{2n+1} - a_1.$$
So it converges. So the second series converges. Converges if the second series converges, the partial sum converges. So S_{2n+1} converges. Since $S_{2n+2} = S_{2n+1} + a_{2n+2}$ and $\lim_{n \to \infty} a_{2n+2} = 0$,
$$\lim_{n \to \infty} S_{2n+2} = \lim_{n \to \infty} S_{2n+1}.$$
So the first series converges. $\square$

DEPARTMENT OF MATHEMATICS, KU, LAWRENCE, KS 66045

E-mail address: slshao@math.ku.edu