Lecture 1

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1.Introduction to systems of linear equations

2.How many solutions to a linear system.

3. Augmented matrices and elementary row operations

Definition of linear equations in n variables

The linear equation in n variables:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where

$$x_1, x_2, \cdots, x_n : \text{ variables;}$$
$$a_1, a_2, \cdots, a_n : \text{ coefficients;}$$
$$b \text{ is a real number.}$$

Note that the powers of x_i are of 1 for $1 \leq i \leq n$.

Examples.

(1). Examples of linear Equations.

$$x + 3y = 7,$$
$$\frac{1}{2}x - y + 3z = -1,$$
$$x_1 - 2x_2 - 3x_3 + x_4 = 0,$$
$$x_1 + x_2 + \cdots + x_4 = 1.$$

(2). Examples of non-linear equations.

$$x + 3y^2 = 4, \text{ involving } y^2,$$
$$3x + 2y - xy = 4, \text{ involving } xy,$$
$$\sin x + y = 0, \text{ involving } \sin x,$$
$$\sqrt{x_1} + 2x_2 + x_3 = 1, \text{ involving } \sqrt{x_1}.$$

System of linear equations.

A linear system of m equations in n unknowns $x_1, x_2, \cdots, x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2, \\ \cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m. \end{cases}$$

Here m denotes the number of linear equations; n denotes the numbers of unknowns.

Examples.

A linear system of 2 unknows.

$$\begin{cases} 5x + y & = 3, \\ 2x - y & = 4. \end{cases}$$

In this example, $m = 2$.
A linear system of 3 unknowns.

$$\begin{cases} 4x_1 - x_2 + 3x_3 & = -1, \\ 3x_1 + x_2 + 9x_3 & = -4 \\ 5x_1 + 2x_2 + 3x_3 & = 0. \end{cases}$$

In this example, $m = 3$.

Definition of solutions to linear systems.

Recall

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2, \\ \dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m. \end{cases}$$

Definition. A solution of a linear system in n unknowns $x_1, x_2, \cdots, x_n$ is a sequence of numbers $s_1, s_2, \cdots, s_n$ such that the substitution

$$x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$$

makes each equation a true statement.

Examples.

1. For $x + y = 0$, $x = 1, y = -1$ is a solution. In general, let $x = t$, then

$$y = -t.$$

Thus the ordered pair $(x, y) = (t, -t)$, $t \in \mathbb{R}$, is a solution. Note that in this example, we have introduced an unknown t, which is called **a parameter.** This way of expressing solutions is called writing the solution in the parameter form.

2. For $x + y + z = 0$, the solution can be expressed in the following way: let $y = s$ and $z = t$, then

$$x = -s - t.$$

Thus

$$(x, y, z) = (-s - t, s, t).$$

In this example, we have introduced two parameters, s and t, and writing x in terms of s, t.

3. For the linear system $\begin{cases} x + y & = 1, \\ x - y & = 2, \end{cases}$, the ordered pair $(x, y) = (3/2, -1/2)$ is a solution.

4. For the linear system $\begin{cases} x + y + z & = 1, \\ x - y - z & = 2, \\ y + 2z & = 0, \end{cases}$, the ordered triple $(x, y, z) = (3/2, -1, 1/2)$ is a solution.

Consistent systems & Inconsistent systems.

1. A linear system is consistent if it has at least one solution.

2. A linear system is inconsistent if it has no solutions.

Example 1. The examples on the previous slide are consistent.

Example 2. The example of linear system $\begin{cases} x+y &= 1, \\ x+y &= 2, \end{cases}$ is inconsistent: no pairs (x, y) make both equations true.

Theorem.

Theorem

Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibility.

Proof postponed.

A linear system with one solution.

$$\begin{cases} x - y &= 1, \\ 2x + y &= 6. \end{cases}$$

Solution. We add two equations together.

$$3x = 7.$$

So

$$x = \frac{7}{3}.$$

Plug this x into $x - y = 1$, we have

$$y = \frac{4}{3}.$$

Thus $(x, y) = (\frac{7}{3}, \frac{4}{3})$ is a solution to this linear system.

A linear system with no solutions.

$$\begin{cases} x + y &= 4, \\ 3x + 3y &= 6. \end{cases}$$

We simplify the second equation,

$$x + y = 2.$$

Compared with $x + y = 4$, obviously there is no (x, y) making both equations true. In this case, this is a linear system with no solutions.

A linear system with infinitely many solutions.

$$\begin{cases} 4x - 2y &= 1, \\ 16x - 8y &= 4. \end{cases}$$

We simplify both equations to obtain

$$\begin{cases} 2x - y &= \frac{1}{2}, \\ 2x - y &= \frac{1}{2}. \end{cases}$$

The two equations coincide. This is one equation with 2 unknowns. So if $x = t$, $y = 2t - \frac{1}{2}$. So

$$(x, y) = (t, 2t - \frac{1}{2})$$

varies as t varies. So there are infinitely many solutions to this system. Note that we have written the solutions in the parameter form.

Augmented matrix.

Recall

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2, \\ \dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m. \end{cases}$$

The augmented matrix is a rectangular array of matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

Examples.

The augmented matrix for the linear system of equations

$$\begin{cases} 4x_1 - x_2 + 3x_3 & = -1, \\ 3x_1 + x_2 + 9x_3 & = -4 \\ 5x_1 + 2x_2 + 3x_3 & = 0. \end{cases} \tag{2}$$

is

$$\begin{bmatrix} 4 & -1 & 3 & -1 \\ 3 & 1 & 9 & -4 \\ 5 & 2 & 3 & 0 \end{bmatrix}.$$

Algebraic operations of linear systems of equations.

Solutions. Multiplying the 1st by 3, and 3rd equation by 3 to obtain

$$\begin{cases} 12x_1 - 3x_2 + 9x_3 & = -3, \\ 3x_1 + x_2 + 9x_3 & = -4 \\ 15x_1 + 6x_2 + 9x_3 & = 0. \end{cases} \tag{3}$$

Subtracting the 2nd equation from the 1st equation, and subtracting 2nd from 3rd to obtain

$$\begin{cases} 9x_1 - 4x_2 & = 1, \\ 12x_1 + 5x_2 & = 4. \end{cases} \tag{4}$$

cont.

Multiplying the new 1st by 5, the new 2nd by 4,

$$\begin{cases} 45x_1 - 20x_2 &= 5, \\ 48x_1 + 20x_2 &= 16. \end{cases} \tag{5}$$

Adding these two equations,

$$93x_1 = 21, \Rightarrow x_1 = \frac{7}{31}.$$

So $x_2 = \frac{11}{18}$ and $x_3 = -\frac{17}{31}$. So the ordered triple $(x_1, x_2, x_3) = (\frac{7}{31}, \frac{11}{18}, -\frac{17}{31})$ is a solution.

To conclude.

Three algebraic operations of equations.

1. Multiplying an equation through by a nonzero constant.

2. Inerchange two equations.

3. Add a constant times one equation to another.

In terms of the augmented matrix,

Definition (Elementary Row Operations.)

1. Multiplying a row through by a nonzero constant.

2. Interchange two rows.

3. Add a constant times one row to another.

Using elementary row operations to solve a linear system.

The augmented matrix for the linear system of equations

$$\begin{cases} x + y + 2z & = 9, \\ 2x + 4y - 3z & = 1 \\ 3x + 6y - 5z & = 0. \end{cases} \tag{6}$$

is

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}.$$

Elementary row operations.

Adding -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}.$$

Adding -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}.$$

Elementary row operations (Cont.)

Multiplying the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}.$$

Adding -3 times the second row to the third and multiply the third row by 2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Elementary row operations (Cont.)

Now we work upwards. Adding $\frac{7}{2}$ times the third row to the second and -2 times the third row to the first.

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Adding -1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Thus $(x, y, z) = (1, 2, 3)$ is the solution the linear system.

Remark. The method of solving the linear system (6) is called "Gaussian Elimination", which is the topic of the next lecture.

Homework and Reading.

Homework. Ex. #2. #4. #9. #12. #13, and the True-False exercise on page 10.

Reading. Section 1.2.