Lecture 10

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Determinants along cofactor expansions

Recall that, if A be a square matrix, then we expand it along the i-th row,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in};$$

if we expand it along the j-th column,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Theorem.

Theorem. Let A be a square matrix. If A has rows of zeros or a column of zeros, then $\det(A) = 0$.

Proof. We expand it along the row or column of zeros and denote the corresponding cofactor $C_1, C_2, \cdots, C_n$. Then

$$\det(A) = 0 \times C_1 + 0 \times C_2 + \cdots + 0 \times C_n = 0.$$

Theorem. Let A be a square matrix. Then

$$\det(A) = \det(A^T).$$

Proof. We prove it by mathematical induction. Suppose A is a matrix of size $n \times n$.

Step 1. $n = 1$. It is obvious that $\det(A) = \det(A^T)$ because

$$A = A^T.$$

Step 2. Suppose the conclusion is true for any matrix of size $n \times n$. We need to prove it for any matrix of size $(n+1) \times (n+1)$. We expand it along the i-th row,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{i(n+1)}C_{i(n+1)};$$

After transposing A, the i-th row becomes the i-th column, so if we expand it along the i-th column,

$$\det(A^T) = a_{1i}C_{1i} + a_{2i}C_{2i} + \cdots + a_{(n+1)i}C_{(n+1)i}.$$

Note that for fixed i, for any $1 \leq j \leq n+1$,

$$C_{ij} = (-1)^{i+j} M_{ij},\ C_{ji} = (-1)^{i+j} M_{ji}.$$

Both M_{ij} and M_{ji} are determinants of matrices of size $n \times n$, and these two matrices are transpose to each other. So by the induction hypothesis,

$$M_{ij} = M_{ji},\ \Rightarrow C_{ij} = C_{ji}.$$

Thus for A of size $(n+1) \times (n+1)$,

$$\det(A) = \det(A^T).$$

Determinants under the elementary row operations.

Theorem. Let A be an $n \times n$ matrix.

(a). If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k \det(A)$.

(b). If B is the matrix that results when two rows or two columns of A are exchanged, then $\det(B) = -\det(A)$.

(c). If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then $\det(B) = \det(A)$.

Proof. (a) Suppose B is obtained from A by multiplying the i-th row by k. Then

$$B = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where the rest entires are same as those in matrix A. Then if we expand the matrix B along the i-th row of A,

$$\begin{aligned} \det(B) &= ka_{i1}C_{i1} + ka_{i2}C_{i2} + \cdots + ka_{in}C_{in} \\ &= k\left(a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}\right) \\ &= k\det(A). \end{aligned}$$

Proof. (b) Suppose that B is obtained from A by exchanging the nearby two rows, the i-th row and the $i+1$-th row of A. Then

$$B = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where the rest entires are the same as those in matrix A. Then we expand the determinant along the $(i+1)$-th row,

$$\det(B) = a_{i1}C_{(i+1)1} + a_{i2}C_{(i+1)2} + \cdots + a_{in}C_{(i+1)n}.$$

Fixing i, for any $1 \leq j \leq n$, we observe that the minor $M_{(i+1)j}$ of B is the same as M_{ij} of A. So $C_{(i+1)j} = (-1)^{i+1+j} M_{(i+1)j} = -(-1)^{i+j} M_{ij} = -C_{ij}$ and thus

$$\begin{aligned}
\det(B) &= a_{i1} C_{(i+1)1} + a_{i2} C_{(i+1)2} + \cdots + a_{in} C_{(i+1)n} \\
&= -a_{i1} C_{i1} - a_{i2} C_{i2} - \cdots - a_{in} C_{in} \\
&= -\left(a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}\right) \\
&= -\det(A).
\end{aligned}$$

In general, we suppose $i < j$. From the previous discussion, the sign changes once two nearby rows are exchanged. So when the i-th row and the j-th row are exchanged, the sign change is

$$\det(B) = (-1)^{2(j-i)-1} \det(A) = -\det(A).$$

To prove **(c)**, we first prove that the determinant of A is zero when it contains two identical rows. This follows from **(b)**. When these two identical rows are exchanged, the resulting matrix is B and

$$\det(B) = -\det(A).$$

However $B = A$. Hence

$$\det(A) = 0.$$

If k times the j-th row is added to the i-th row, we get

$$B = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{i1} + ka_{j1} & a_{i2} + ka_{j2} & \cdots & a_{in} + ka_{jn} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

We expand $\det(B)$ along the i-th row, by **(a)** and **(b)**,

$$|B| = \begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} + k \begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$= \det(A) + 0 = \det(A).$$

So the third elementary row operation does not change the determinant of matrices.

Recall that an elementary matrix is obtained from I_n when a single elementary row operation is performed. By the previous theorem,

Theorem. Let E be an $n \times n$ elementary matrix.

(a). If E results from multiplying a row of I_n by a nonzero number k, then

$$\det(E) = k.$$

(b). If E results from interchanging two rows of I_n, then

$$\det(E) = -1.$$

(c). If E results from adding a multiple of one row of I_n to another, then

$$\det(E) = 1.$$

Theorem. If A is a square matrix with two proportional rows or two proportional columns, then

$$\det(A) = 0.$$

Example. (a).

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3, \text{ 3 times the second row.}$$

(b).

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1, \text{ exchange the 1st row and the 4th row.}$$

(c).

$$\begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1. \text{ 7 times the 4th row added to the 1st row.}$$

Using row reduction to evaluate a determinant.

Evaluate $\det(A)$ where $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$.

Solution. We compute

$$\det(A) = \begin{matrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{matrix} = - \begin{matrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{matrix}$$

$$= -3 \begin{matrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{matrix} = -3 \begin{matrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{matrix}$$

$$= -3 \begin{matrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{matrix} = -3 \times 1 \times 1 \times (-55) = 165.$$

Using column operation to evaluate a determinant.

Compute the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$.

Solution. Adding -3 times the first column to the fourth column to obtain

$$\det(A) = \begin{matrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{matrix} = 1 \times 7 \times 3 \times (-26) = -546.$$

Cofactor expansion and row operations

$$\det(A) = \begin{matrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{matrix} \quad \text{cofactor expansion along the first column}$$

$$= - \begin{matrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{matrix} \quad \text{adding the first row to the third row}$$

$$= - \begin{matrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{matrix} \quad \text{adding -3 times 2nd row to the third row}$$

$$= - \begin{matrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & -6 \end{matrix} = -18.$$

Homework and Reading.

Homework. Ex. #2, #6,#8, #12, #15, #20, #24,#26, #29, and the True-False exercise on page 106.

Reading. Section 2.3.