Lecture 11

Shuanglin Shao

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Matrix determinants: addition.

Determinants: multiplication.

Adjoint of a matrix.

Cramer's rule to solve a linear system.

Recall that from the previous section, if a single row of A is multiplied by k, then for the resulting matrix,

$$\det(B) = k \det(A).$$

If the matrix $B = kA$, i.e., every entries in A are multiplied by the constant k, then we iterated the previous theorem

$$\det(B) = k^n \det(A).$$

Example: $\det(A+B) \neq \det(A) + \det(B)$.

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} ?A + B == \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}.$$

We have $\det(A) = 1$, $\det(B) = 8$, and $\det(A+B) = 23$. Thus

$$\det(A+B) \neq \det(A) + \det(B).$$

Theorem. Let A, B and C be $n \times n$ matrices that differ only in a single row, say the r-th row, and assume that the r-th row can be obtained by adding corresponding entries in the r-th row of A and B, then

$$\det(C) = \det(A) + \det(B).$$

This is proved by row expansion of determinants along the r-th row of C. Suppose that

$$A = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}; B = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix},$$

and

$$C = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ a_{r1} + b_{r1} & a_{r2} + b_{r2} & \cdots & a_{rn} + b_{rn} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix},$$

where the rest entires are the same.

We first observe that, if fixing the (r, j)-entry in either A, B or C, then the cofactors are the same, i.e., C_{rj} is identical for A, B and C.

$$\begin{aligned}\det(C) &= \sum_{j=1}^{n}(a_{rj} + b_{rj})C_{rj} \\ &= \sum_{j=1}^{n} a_{rj}C_{rj} + \sum_{j=1}^{n} b_{rj}C_{rj} \\ &= \det(A) + \det(B).\end{aligned}$$

Note that $A + B$ is not the same as C: $A + B$ is obtained by adding the corresponding entries.

Example.

$$\begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{pmatrix} = \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}.$$

Determinants of a product.

Lemma 2.3.2. If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B).$$

Let E be an elementary matrix. If we multiply B by E from the left, EB is the matrix obtained by performing an elementary row operation on B.

Case 1. If E is obtained from I_n by multiplying k to a row, then EB is the matrix that is obtained by multiplying k to the same row.

$$\det(EB) = k\det(B) = \det(E)\det(B)$$

because $\det(E) = k$.

Case 2. If E is obtained from I_n by exchanging two rows, then EB is the matrix that is obtained by exchanging the same two rows:

$$\det(EB) = -\det(B) = \det(E)\det(B)$$

because $\det(E) = -1$.

Case 3. If E is obtained from I_n by adding k times a row to another row, then EB is the matrix that is obtained by adding k times the same row to the same another row:

$$\det(EB) = \det(B) = \det(E)\det(B)$$

because $\det(E) = 1$.

Remark. If B is an $n \times n$ matrix and $E_1, E_2, \cdots, E_r$ are $n \times n$ elementary matrices, then

$$\det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B).$$

Determinant test for invertibility

Theorem. A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof. If A is invertible, then A is expressed as a product of elementary matrices:

$$A = E_1 E_2 \cdots E_r.$$

Then

$$\det(A) = \det(E_1)\det(E_2)\cdots\det(E_r).$$

Since for each elementary matrix E, $\det(E) \neq 0$. Thus

$$\det(A) \neq 0.$$

Cont.

On the other hand, suppose $\det(A) \neq 0$. We apply elementary row operations to reduce A to reduced row echelon form R. That is to say,

$$R = E_1 E_2 \cdots E_r A.$$

Then

$$\det(R) = \det(E_1) \det(E_2) \cdots \det(E_r) \det(A) \neq 0.$$

This implies

$$\det(R) \neq 0.$$

Since R is a square matrix and is in reduced row echelon form, R is the identity matrix. In other words, A reduces to I_n after a series of elementary row operations. Hence A is invertible.

Example. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$. Since the first and third rows of A is proportional,

$$\det(A) = 0.$$

Hence A is not invertible.

If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B).$$

Case 1. If A is not invertible, $\det(A) = 0$. It also follows that AB is not invertible, which implies that $\det(AB) = 0$. Thus

$$\det(AB) = 0 = \det(A)\,\mathrm{B}\,.$$

Case 2. If A is invertible, then A is a product of elementary matrices, $E_1, E_2, \cdots, E_r$:

$$A = E_1 E_2 \cdots E_r B.$$

Thus

$$\det(AB) = \det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B).$$

Since $\det(A) = \det(E_1)\det(E_2)\cdots\det(E_r)$,

$$\det(AB) = \det(A)\det(B).$$

Consider $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$.
The product

$$AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}.$$

We verify that

$$\det(A) = 1,\ \det(B) = -23, \det(AB) = -23.$$

Thus

$$\det(AB) = \det(A)\det(B).$$

A corollary to the theorem is

Theorem. If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

This is because

$$I_n = A \times A^{-1}.$$

Thus

$$1 = \det(I_n) = \det(A)\det(A^{-1}).$$

Def. If A is any $n \times n$ matrix, and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

is called the adjoint of the matrix A, denoted by $\text{adj}(A)$.

Recall that $C_{ij} = (-1)^{i+j} M_{ij}$, and M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by removing the i-th row and the j-th column of A.

Example.

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}.$$

The adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.$$

We compute

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = 4.$$

Inverse of a matrix using its adjoint.

Theorem. If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A).$$

Proof. We show that

$$A\,\text{adj}(A) = \det(A) I_n.$$

Let $B = A\,\text{adj}(A) = [b_{ij}]$. The (i,j)-entry b_{ij} of the product matrix $A\,\text{adj}(A)$ is coming from the i-th row of A and the j-th column of $\text{adj}(A)$. Note that the j-th column of $\text{adj}(A)$ is the cofactor of the entry a_{jk} of the matrix A, $1 \le k \le n$.

$$b_{ij} = \sum_{k=1}^{n} a_{ik} C_{jk} = a_{i1} C_{j1} + a_{i2} C_{j2} + \cdots + a_{in} C_{jn}.$$

Fixing i. We discuss two cases.

Case 1. When $j = i$, by the definition of determinants of matrices,

$$b_{ij} = \det(A).$$

Case 2. When $j \neq i$, we prove that $b_{ij} = 0$. Suppose that $i < j$; the proof for $i > j$ is similar. We construct a matrix that differs from the matrix A only in the j-th row: the j-th row is the same as the i-th row.

$$D = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}.$$

Since D contains two identical rows, $\det(D) = 0$. We expand the determinant along the j-th row:

$$\det(D) = 0 = a_{i1}C'_{j1} + a_{i2}C'_{j2} + \cdots a_{in}C'_{jn}.$$

Since D only differs from A in the j-th row, $C'_{jk} = C_{jk}$. So

$$b_{ij} = 0, \text{ for } i \neq j.$$

Hence the matrix B is $\det(A)I_n$. This proves the theorem.

Using the adjoint to find an inverse matrix.

Example. $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$. From the previous example, the adjoint of A is computed. By the theorem, the inverse of A is

$$\frac{1}{\det(A)} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.$$

The determinant of A, $\det(A) = 64$. So the inverse is

$$\begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}.$$

Cramer's rule.

Theorem. If $Ax = b$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \cdots, x_n = \frac{\det(A_n)}{\det(A)}$$

when A_j is the matrix obtained by replacing the entries in the j-th column of A by the entires in the matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If $\det(A) \neq 0$, then A is invertible. We multiply $Ax = b$ by A^{-1} to obtain

$$x = A^{-1}b = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \dots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

We multiply the matrices on the right hand side out to obtain the i-th entry.

$$b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni}.$$

This is precisely the determinant of the following matrix expanding along the i-th column:

$$\begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots\vdots & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

where the i-th column of A is replaced by the column vector. So

$$x_i = \frac{\det(A_i)}{\det(A)}.$$

Using Cramer's rule to solve a linear system.

Use the Cramer's rule to solve

$$\begin{cases} x_1 + 2x_3 & = 6, \\ -3x_1 + 4x_2 + 6x_3 & = 30, \\ -x_1 - 2x_2 + 3x_3 & = 8. \end{cases}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

The complexity here is to compute the determinants here.

Therefore

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-10}{11},\ x_2 = \frac{\det(A_2)}{\det(A)} = \frac{18}{11}.$$

and

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{38}{11}.$$

We usually use the elementary row operations to compute the determinants of matrices.

Homework and Reading.

Homework. Ex. #4, #6,#8, #10, #16, #18, #20,#24, #30, and the True-False exercise on page 116.

Reading. Section 3.1.