Lecture 12: Section 3.1

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Geometric vectors

We represent vectors in two dimensions (2-spaces) or in three dimensions (3 -spaces) by **arrows**. The **direction** of the arrowhead specifies the direction of the vector and the **length** of the arrow specifies the magnitude. The tail of the the arrow is called the **initial** point of the vector and the tip the **terminal point.**

In this text, we will denote vectors in boldface type such as **a, b, v, w, x** and we will denote scalars in lower italic type such as $a, k, v, w,$ and x.

$$\mathbf{v} = \overrightarrow{AB}$$

denotes the vector with the initial point A and the terminal point B.

1. Vectors with the same length and the same direction are said to be **equivalent**:

$$\mathbf{v} = \mathbf{w}.$$

2. **Zero vector** : the vector whose initial and terminal points coincide has length zero. We denote it by $\mathbf{0}$.

Vector addition: 2 vectors

Parallelogram rule for vector addition. If $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincide, then the two vectors form adjacent sides of a parallelogram, and the **sum** $\mathbf{v} + \mathbf{w}$ is the vector represented by the arrow from the common initial point of $\mathbf{v}$ and $\mathbf{w}$ to the opposite vertex of the parallelogram.

Vector subtraction: 2 vectors

The **negative** of a vector $\mathbf{v}$, denoted by $-\mathbf{v}$, is the vector that has the same length as $\mathbf{v}$ but is oppositely directed, and the difference of $\mathbf{v}$ from $\mathbf{w}$, denoted by $\mathbf{w} - \mathbf{v}$, is taken to be the sum

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}).$$

scalar multiplication: $k\mathbf{v}$.

If $\mathbf{v}$ is a nonzero vector in 2-space or 3-space, and if k is a nonzero scalar, then we define the scalar product of $\mathbf{v}$ by $\mathbf{k}$ to be the vector whose length is $|k|$ times the length of $\mathbf{v}$, and whose direction is the same as that of $\mathbf{v}$ and if k is positive and opposite to that of $\mathbf{v}$ if k is negative. If $k = 0$ and $\mathbf{v} = 0$, then we define $k\mathbf{v}$ to be $\mathbf{0}$.

collinear and parallel vectors

Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2-space or 3-space with a common initial point. If one of the vectors is a scalar multiple of the other, then the vectors lie on a common line. In this case, we say that $\mathbf{v}$ and $\mathbf{w}$ are **collinear**.

If we translate one of the vectors, then the vectors are **parallel** but no longer collinear.

Sums of three or more vectors

If we have three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$,

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

This can be proven by repeatedly using the parallelogram rule for vector addition.

Since there is no difference by grouping the first two, or the latter two vectors, we denote by

$$\mathbf{u} + \mathbf{v} + \mathbf{w}$$

the vector $\mathbf{u} + (\mathbf{v} + \mathbf{w})$ or $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

Vectors in coordinate system: n-space.

Definition. If n is a positive integer, an **ordered n-tuple** is a sequence of n real numbers $(v_1, v_2, \cdots, v_n)$. The set of all ordered n-tuples is called n-space and is denoted by $\mathbb{R}^n$. We denote

$$\mathbf{v} = (v_1, v_2, \cdots, v_n),$$

or the zero vector

$$\mathbf{0} = (0, 0, \cdots, 0).$$

Example. When $n = 2$, the Euclidean plane.
When $n = 3$, the Euclidean space.

Equality of vectors.

Definition. Vectors $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ and $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ in $\mathbb{R}^n$ are said to be **equivalent** or **equal** if

$$v_1 = w_1, v_2 = w_2, \cdots, v_n = w_n.$$

We indicate this by writing

$$\mathbf{v} = \mathbf{w}.$$

Example: Equality of vectors.

$$(a, b, c, d) = (1, -4, 2, 7)$$

if and only if

$$a = -1, b = -4, c = 2, d = 7.$$

Definition. If $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ and $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ are vectors in $\mathbb{R}^n$ and if k is any scalar, then we define

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n), \\
k\mathbf{v} &= (kv_1, kv_2, \cdots, kv_n), \\
-\mathbf{v} &= (-v_1, -v_2, \cdots, -v_n), \\
\mathbf{w} - \mathbf{v} &= \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \cdots, w_n - v_n).
\end{aligned}$$

Algebraic operations using components.

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= (5, -1, 3), \\
2\mathbf{v} &= (2, -6, 4), \\
-\mathbf{w} &= (-4, -2, -1), \\
\mathbf{v} - \mathbf{w} &= \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1).
\end{aligned}$$

Theorem 3.1.1. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and if k, m are scalars, then

$$
\begin{aligned}
\mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u}, \\
(\mathbf{u}+\mathbf{v})+\mathbf{w} &= \mathbf{u}+(\mathbf{v}+\mathbf{w}), \\
\mathbf{u}+\mathbf{0} &= \mathbf{0}+\mathbf{u} = u, \\
\mathbf{u}+(-\mathbf{u}) &= 0, \\
\mathbf{k}(\mathbf{u}+\mathbf{v}) &= k\mathbf{u}+k\mathbf{v}, \\
(k+m)\mathbf{u} &= k\mathbf{u}+m\mathbf{u} \\
k(m\mathbf{u}) &= (km)\mathbf{u}, \\
1\mathbf{u} &= \mathbf{u}.
\end{aligned}
$$

This is proven by using the component representation of vectors in $\mathbb{R}^n$.

Proof of the first identity in the Theorem above.

Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$, then by definition of vector addition in $\mathbb{R}^n$,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n).$$

Also

$$\mathbf{v} + \mathbf{u} = (v_1 + u_1, v_2 + u_2, \cdots, v_n + u_n).$$

As real numbers,

$$u_j + v_j = v_j + u_j,\ 1 \leq j \leq n.$$

We have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

Theorem 3.1.2. If $\mathbf{v}$ is a vector in $\mathbb{R}^n$ and k is a scalar, then

$$0\mathbf{v} = \mathbf{0},$$
$$k\mathbf{0} = \mathbf{0},$$
$$(-1)\mathbf{v} = -\mathbf{v}.$$

This is proven by using the component representation of vectors in $\mathbb{R}^n$

Definition. If $\mathbf{w}$ is a vector in $\mathbb{R}^n$, then $\mathbf{w}$ is said to be a linear combination of the vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}$ in $\mathbb{R}^n$ if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_r\mathbf{v_r},$$

where $k_1, k_2, \cdots, k_r$ are scalars. These scalars are called the **coefficients** of the linear combination.

Example.

Find all scalars c_1, c_2 and c_3 such that

$$c_1(1,2,0) + c_2(2,1,1) + c_3(0,3,1) = (0,0,0).$$

Solution.

$$\textbf{LHS} = (c_1 + 2c_2, 2c_1 + c_2 + 3c_3, c_2 + c_3).$$

By equating it to the right hand side, we have the following linear system of equations:

$$\begin{cases} c_1 + 2c_2 & = 0, \\ 2c_1 + c_2 + 3c_3 & = 0, \\ c_2 + c_3 & = 0. \end{cases}$$

The argument matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

So

$$c_1 = 0, c_2 = 0, c_3 = 0.$$

(a). Exercises: # 6,#8, # 9, # 10, # 12,# 14, (b), (d), (f), # 22, # 24,# 28, True or False questions on page 130.

(b). Read Section 3.2.