Lecture 13: Section 3.2

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Definition. If $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is a vector in $\mathbb{R}^n$, then the **norm** of $\mathbf{v}$ (also called the **length** of $\mathbf{v}$ or the **magnitude** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Example 1.

Let $\mathbf{v} = (-3, 2, 1)$. Then

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}.$$

Let $\mathbf{v} = (2, -1, 3, -5)$. Then

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39}.$$

Theorem If $\mathbf{v}$ is a vector in $\mathbb{R}^n$ and if k is any scalar, then

(a). $\|\mathbf{v}\| \geq 0$.

(b). $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$.

(c). $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.

Proof.

We prove the part **(c)**. Let $\mathbf{v} = (v_1, v_2, \cdots, v_n)$. Then

$$k\mathbf{v} = (kv_1, kv_2, \cdots, kv_n).$$

Hence

$$\begin{aligned} \|k\mathbf{v}\| &= \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2} \\ &= |k|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\ &= |k|\|\mathbf{v}\|. \end{aligned}$$

□

Unit Vectors.

If $\mathbf{v}$ is any nonzero vector in $\mathbb{R}^n$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$ defines a unit vector that is in the same direction as $\mathbf{v}$.

Example. Find the unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution. The vector $\mathbf{v}$ has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

Thus

$$\mathbf{u} = \frac{v}{\|\mathbf{v}\|} = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}).$$

The standard unit vectors.

(1). $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are the standard unit vectors in $\mathbb{R}^2$.

(2). $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $k = (0, 0, 1)$ are the standard unit vectors in $\mathbb{R}^3$.

Any vector in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be expressed as linear combinations of $\mathbf{i}, \mathbf{j}$ or $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

$$(v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j},$$
$$(v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

These standard unit vectors can be generalized to be those in $\mathbb{R}^n$:

$$\mathbf{e}_1 = (1, 0, 0 \cdots, 0),\ \mathbf{e}_2 = (0, 1, 0, \cdots, 0),\ \mathbf{e}_n = (0, 0, \cdots, 1),$$

In which case, every vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ can be expressed in terms of a linear combination,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

Linear combinations of standard unit vectors.

$$(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k},$$
$$(7, 3, -4, 5) = 7\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3 + 5\mathbf{e}_4.$$

Distance in $\mathbb{R}^n$.

Definition. If $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ are points in $\mathbb{R}^n$, then we denote the distance between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

Calculating distance in $\mathbb{R}^n$.

If $\mathbf{u} = (1, 3, -2, 7)$ and $\mathbf{v} = (0, 7, 2, 2)$, then the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}.$$

Definition. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ are vectors in $\mathbb{R}^n$, then the **dot product** (also called the Euclidean inner product) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Calculating dot product using components.

Example. Let $\mathbf{u} = (-1, 3, 5, 7)$ and $\mathbf{v} = (-3, -4, 1, 0)$. Then

$$\mathbf{u} \cdot \mathbf{v} = -1 \times (-3) + 3 \times (-4) + 5 \times 1 + 7 \times 0 = -4.$$

Example: inner product in $\mathbb{R}^2$ and $\mathbb{R}^3$.

When $n = 2$ or $n = 3$, the Euclidean inner product $\mathbf{u}, \mathbf{v}$ can be written as a different form, which involves the angle of the vectors $\mathbf{u}, \mathbf{v}$.

Suppose that $\mathbf{u}, \mathbf{v}$ are vectors with the starting points at the origin. Then $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ forms a triangle. By the law of cosines, we have

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and $0 \leq \theta \leq \pi$.

Suppose that $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2, \\ \|\mathbf{u}\|^2 &= u_1^2 + u_2^2 + u_3^2, \\ \|\mathbf{v}\|^2 &= v_1^2 + v_2^2 + v_3^2. \end{aligned}$$

Thus

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta.$$

Therefore the inner product in $\mathbb{R}^3$ can take the following form

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta.$$

If $\mathbf{u} \neq 0$ and $\mathbf{v} \neq 0$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

Since $0 \leq \theta \leq \pi$, it follows that

1. If $\mathbf{u} \cdot \mathbf{v} > 0$, then θ is acute, $0 < \theta < \frac{\pi}{2}$.
2. If $\mathbf{u} \cdot \mathbf{v} < 0$, then θ is obtuse, $\frac{\pi}{2} < \theta < \pi$.
3. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\theta = \frac{\pi}{2}$.

Example.

Let $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$. Then

$$\|\mathbf{u}\| = 1, \ \|\mathbf{v}\| = 2\sqrt{2}.$$

and

$$\mathbf{u} \cdot \mathbf{v} = 0 \times 0 + 0 \times 2 + 1 \times 2 = 2.$$

Thus

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}.$$

Since $0 \leq \theta \leq \pi$,

$$\theta = \frac{\pi}{4}.$$

Algebraic properties of the dot product.

Specify **u** and **v** in Definition, then

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2.$$

Thus

$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|.$$

Theorem. If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and if k is a scalar, then

(a). $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(b). $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

(c). $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$.

(d). $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = 0$.

(c). Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$. Then

$$\begin{aligned} k(\mathbf{u} \cdot \mathbf{v}) &= k(u_1 v_1 + u_2 v_2 + \cdots + u_n v_n) \\ &= (ku_1)v_1 + (ku_2)v_2 + \cdots + (ku_n)v_n \\ &= (k\mathbf{u}) \cdot \mathbf{v}. \end{aligned}$$

(d).

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2.$$

Thus $\mathbf{v} \cdot \mathbf{v} \geq 0$. Also

$$\mathbf{v} \cdot \mathbf{v} = 0 \Leftrightarrow v_j = 0 \text{ for each } j \Leftrightarrow \mathbf{v} = 0.$$

Theorem. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, and if k is a scalar, then

(a). $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$.

(b). $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

(c). $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$.

(d). $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$.

(e). $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$.

This can be proven by using the components of vectors.

(a). Let $\mathbf{0} = (0, 0, \cdot, 0)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$. Thus

$$\mathbf{0} \cdot \mathbf{v} = 0 \times v_1 + 0 \times v_2 + \cdots + 0 \times v_n = 0.$$

Similarly for $\mathbf{v} \cdot \mathbf{0}$; and $(\mathbf{b}) - - - (\mathbf{e})$.

Calculating Dot products.

$$\begin{aligned}(\mathbf{u} - \mathbf{2v}) \cdot (3\mathbf{u} + 4\mathbf{v}) &= \mathbf{u} \cdot (3\mathbf{u} + 4\mathbf{v}) - 2\mathbf{v} \cdot (3\mathbf{u} + 4\mathbf{v}) \\ &= 3(\mathbf{u} \cdot \mathbf{u}) + 4\mathbf{u} \cdot \mathbf{v} - 6\mathbf{v} \cdot \mathbf{u} - 8\mathbf{v} \cdot \mathbf{v} \\ &= 3\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 8\|\mathbf{v}\|^2.\end{aligned}$$

The Cauchy- Schwarz inequality

If $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ are vectors in $\mathbb{R}^n$, then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

or in terms of components

$$|u_1 v_1 + u_2 v_2 + \cdots u_n v_n| \leq \left(u_1^2 + \cdots + u_n^2\right)^{1/2} \left(v_1^2 + \cdots + v_n^2\right)^{1/2}.$$

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\| &\leq \|\mathbf{u}\|+\|\mathbf{v}\|, \\ d(\mathbf{u},\mathbf{v}) &\leq d(\mathbf{u},\mathbf{w})+d(\mathbf{v},\mathbf{w}). \end{aligned}$$

Both are called the triangle inequality.

(a).

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &\le (\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u},\mathbf{v}) + \mathbf{v},\mathbf{v}\\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2\\ &= \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2\\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.\end{aligned}$$

Thus

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

(b).

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \|(\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v})\| \\ &\leq \|\mathbf{u} - \mathbf{w}\| + \|\mathbf{w} - \mathbf{v}\| \\ &= d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v}). \end{aligned}$$

Parallelogram equation for vectors.

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2).$$

Proof.

$$\begin{aligned}
&\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 \\
&= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\
&= 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2).
\end{aligned}$$

Theorem. if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2.$$

Proof.

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2,$$
$$\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2.$$

Subtracting these two identities, we have

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2.$$

□

Dot Product as Matrix Multiplication.

Let **u** and **v** are column vectors in $\mathbb{R}^n$. We write the elements in $\mathbb{R}^n$ as column vectors.

Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

and

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Then by using the inner products in the Euclidean space $\mathbb{R}^n$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

As matrix multiplication,

$$\mathbf{u}^T\mathbf{v} = \left[\ u_1, u_2, \cdots, u_n\ \right] \times \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Similarly for

$$\mathbf{v}^T\mathbf{u}.$$

Hence we have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}^T.$$

If A is an $n \times n$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices, then we have

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}.$$
$$\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v}.$$

Example: verifying that $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$.

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}.$$

Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix},$$

and

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

and

$$\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

$$A\mathbf{u} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}.$$

and

$$A^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

Thus

$$A\mathbf{u} \cdot \mathbf{v} = 7(-2) + 10 \times 0 + 5 \times 5 = 11.$$

and

$$\mathbf{u} \cdot A^T\mathbf{v} = (-1)(-7) + 2 \times 4 + 4 \times (-1) = 11.$$

Therefore $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$.

Homework and Reading.

Homework. Exercise. # 4, # 6, # 8, # 12, #19 (a) (c), # 20, (a) (c), # 26 (a) (b), # 27. True or false questions on page 143.

Reading. Section 3.3.