# Lecture 14: Section 3.3

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**Definition.** Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$. A nonempty set of vectors in $\mathbb{R}^n$ is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an **orthonormal set**.

# Example.

**(a).** Show that $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal vectors in $\mathbb{R}^4$.
**Solution.** These two vectors are orthogonal since

$$-2 \times 1 + 3 \times 2 + 1 \times 0 + 4 \times (-1) = 0.$$

**(b).** Show that the set $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of standard unit vectors is an orthogonal set in $\mathbb{R}^3$.
**Solution.** We prove that $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = \mathbf{0}$.

$$\mathbf{i} \cdot \mathbf{j} = (1, 0, 0) \cdot (0, 1, 0) = 0.$$

The equation of lines and planes:

$$\mathbf{n} \cdot \overrightarrow{P_0 P} = 0,$$

where $n$ is the normal vector to the line or the plane, $P_0$ is a point on the line. In particular, if $P = (x, y, z)$ and $P_0 = (x_0, y_0, z_0)$, then

$$\mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

If $n = (a, b, c)$, then

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

# Example.

In $\mathbb{R}^2$ the equation

$$6(x-3)+(y+7)=0$$

represents the line through the point $(3,-7)$ with normal $\mathbf{n}=(6,1)$.

In $\mathbb{R}^3$, the equation

$$4(x-3)+2y-5(z-7)=0$$

represents the plane through the point $(3,0,7)$ with normal $\mathbf{n}=(4,2,-5)$.

**(a).** If $a$ and $b$ are constants that are not both zero, then an equation of the form

$$ax + by + c = 0$$

represents a line in $\mathbb{R}^2$ with normal $\mathbf{n} = (a, b)$.

**(b).** If $a, b$ and $c$ are constants that are not all zero, then an equation of the form

$$ax + by + cz + d = 0$$

represents a plane in $\mathbb{R}^3$ with normal $n = (a, b, c)$.

**Solution.** If $a \neq 0$, then

$$a(x + \frac{c}{a}) + b(y - 0) = 0.$$

This is the line through $(-\frac{c}{a}, 0)$ with the normal $(a, b)$.

The part **(b)** is proven similarly.

# Example.

The equation $\mathbf{ax} + \mathbf{by} = \mathbf{0}$ represents a line through the origin in $\mathbb{R}^2$. Show that the vector $\mathbf{n}_1 = (a, b)$ formed from the coefficients of the equation is orthogonal to the line, that is, orthogonal to every vector along the line.

**Solution.** If $(x, y)$ is a point on the line $ax + by = 0$, then

$$(a, b) \cdot (x, y) = 0.$$

It means that $(a, b)$ is orthogonal to vectors $(x, y)$ along the line.

The equation $\mathbf{ax} + \mathbf{by} + \mathbf{cz} + \mathbf{d} = \mathbf{0}$ represents a plane through the origin in $\mathbb{R}^3$. Show that the vector $\mathbf{n}_2 = (a, b, c)$ formed from the coefficients of the equation is orthogonal to the plane, that is, orthogonal to every vector that lies in the plane.

This is proven similarly.

Recall that $\mathbf{ax} + \mathbf{by} = \mathbf{0}$ and $\mathbf{ax} + \mathbf{by} + \mathbf{cz} = \mathbf{0}$ are called homogeneous equation. They can be written as

$$\mathbf{n} \cdot \mathbf{x} = 0$$

where $\mathbf{n}$ is the vector of coefficients and $\mathbf{x}$ is the vector of unknowns.
In $\mathbb{R}^2$, this is called the **vector form of a line** through the origin, and in $\mathbb{R}^3$, it is called the **vector form of a plane** through the origin.

# Projection Theorem.

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and if $\mathbf{a} \neq 0$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w_1} + \mathbf{w_2}$, where $\mathbf{w_1}$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{w_2}$ is orthogonal to $\mathbf{a}$.

**Solution.**

$$\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} + w_2.$$

**Solution.** We suppose that $\mathbf{u} = k\mathbf{a} + \mathbf{w_2}$, where $\mathbf{w_2}$ is orthogonal to $\mathbf{a}$. We multiply both sides by $\mathbf{a}$. Then

$$\mathbf{u} \cdot \mathbf{a} = k\mathbf{a} \cdot \mathbf{a} + \mathbf{w_2} \cdot \mathbf{a} = k\|\mathbf{a}\|^2.$$

Thus

$$k = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}.$$

Thus

$$\mathbf{u} = k\mathbf{a} + \mathbf{w}_2.$$

The orthogonal projection of **u** on **a**:

$$\operatorname{proj}_{\mathrm{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|a\|^2}\mathbf{a}$$

The vector component of **u** along **a**.

$$\mathbf{u} - \operatorname{proj}_{a} \mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

The vector component of **u** orthogonal to **a**.

# Example: orthogonal projection on a line.

Find the orthogonal projection of the vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ on the line $L$ that makes an angle $\theta$ with the positive $x$-axis in $\mathbb{R}^2$.

**Solution.** $\mathbf{a} = (\cos\theta, \sin\theta)$ is a unit vector along the line $L$ because the line makes an angle $\theta$ with $x$-axis. So our first problem is to find the orthogonal projection of $\mathbf{e_1}$ along $\mathbf{a}$. By the formula,

$$\frac{\mathbf{e_1} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{(1,0)\cdot(\cos\theta, \sin\theta)}{\cos^2\theta + \sin^2\theta}(\cos\theta, \sin\theta) = (\cos^2\theta, \sin\theta\cos\theta).$$

Similarly for $\mathbf{e_2} = (0, 1)$,

$$\frac{\mathbf{e_2} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{(0,1)\cdot(\cos\theta, \sin\theta)}{\cos^2\theta + \sin^2\theta}(\cos\theta, \sin\theta) = (\sin\theta\cos\theta, \sin^2\theta).$$

# Example: vector component of $\mathbf{u}$ along $\mathbf{a}$.

**Example.** Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

**Solution.** We compute

$$\mathbf{u} \cdot \mathbf{a} = 2 \times 4 + (-1) \times (-1) + 3 \times 2 = 15,$$

and

$$\|a\|^2 = 4^2 + (-1)^2 + 2^2 = 21.$$

Then the vector component of $\mathbf{u}$ along $\mathbf{a}$ is

$$\text{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} = \frac{15}{21}(4, -1, 2) = (\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}),$$

and

$$\mathbf{u} - \text{proj}_{\mathbf{a}} \mathbf{u} = (2, -1, 3) - (\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}) = (-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}).$$

**Theorem of Pythagoras in $\mathbb{R}^n$.** If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

**Proof.** The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\mathbf{u}\cdot\mathbf{v} = 0.$$

Thus

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

# Example: Theorem of Pythagoras.

Let $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal. Verify the Theorem of Pythagoras.

**Solution.** We compute

$$\mathbf{u} \cdot \mathbf{v} = -2 \times 1 + 3 \times 2 + 1 \times 0 + 4 \times (-1) = 0.$$

Thus $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Thus the Theorem of Pythogoras holds for these two vectors. We verify

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|(-1, 5, 1, 3)\|^2 = (-1)^2 + 5^2 + 1^2 + 3^2 = 36.$$

and

$$\|\mathbf{u}\|^2 = (-2)^2 + 3^2 + 1^2 + 4^2 = 30, \ \|\mathbf{v}\|^2 = 1^2 + 2^2 + 0^2 + (-1)^2 = 6.$$

Hence

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

# Homework and Reading.

**Homework.** Exercise. # 2, # 4, # 5, # 10, #14, # 28. True or false questions on page 152.

**Reading.** Section 3.4.