

Lecture 15: Section 3.4

Shuanglin Shao

October 28, 2013

In this section we will use **parametric and vector methods** to study general systems of linear equations. This work will enable us to interpret **solution sets of linear systems** with n unknowns as **geometric objects in \mathbb{R}^n** .

Vector and Parametric equations of lines in \mathbb{R}^2 and \mathbb{R}^3 .

Theorem. Let L be the line in \mathbb{R}^2 or \mathbb{R}^3 that contains the point x_0 and is parallel to the nonzero vector \mathbf{v} . Then the equation of the line through x_0 that is parallel to \mathbf{v} is

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}. \quad (1)$$

If $x_0 = \mathbf{0}$, then the line passing through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v}.$$

Example.

Let $x_0 = (1, 2, 3)$ and $\mathbf{v} = (1, 1, 1)$. Then the equation passing through x_0 and \mathbf{v} is

$$x = x_0 + t\mathbf{v},$$

that is to say,

$$x = (1, 2, 3) + t(1, 1, 1).$$

If writing $x = (x_1, x_2, x_3)$, we have

$$x_1 = 1 + t, \quad x_2 = 2 + t, \quad x_3 = 3 + t.$$

This is the parametric form of the line L .

Example.

Find a vector equation and parametric equations of the line in \mathbb{R}^2 that passes through the origin and is parallel to the vector $\mathbf{v} = (2, 3)$.

Solution. Since the line passes through the origin, $x_0 = \mathbf{0}$, and the equation can be expressible in the vector form

$$(x, y) = t(-2, 3).$$

In the parametric form,

$$x = -2t, y = 3t.$$

Find a vector equation and parametric equations of the line in \mathbb{R}^3 that passes through the point $P_0(1, 2, -3)$ and is parallel to the vector $\mathbf{v} = (4, -5, 1)$.

Solution. In the vector form,

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} = (1, 2, -3) + t(4, -5, 1).$$

In the parametric form,

$$\begin{aligned}x &= 1 + 4t, \\y &= 2 - 5t, \\z &= -3 + t.\end{aligned}$$

Vector and Parametric equations of planes in \mathbb{R}^3 .

Let \mathbf{W} be the plane in \mathbb{R}^3 that contains the point x_0 and is parallel to the non-collinear vectors \mathbf{v}_1 and \mathbf{v}_2 . Then an equation of the plane through x_0 that is parallel to \mathbf{v}_1 and \mathbf{v}_2 is given by

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2. \quad (2)$$

If $x_0 = 0$, then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

Example.

Let $x_0 = (2, 1, 3)$, $\mathbf{v}_1 = (1, 0, 1)$ and $\mathbf{v}_2 = (0, 1, 1)$. Find the plane that passes through x_0 and is parallel to the vectors \mathbf{v}_1 and \mathbf{v}_2 .

Solution. The vectors $\mathbf{v}_1, \mathbf{v}_2$ are not collinear because both of them are starting from the origin, and one of them is not a multiple of the other.

By using the theorem above,

$$\begin{aligned}x &= x_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \\&= (2, 1, 3) + t_1(1, 0, 1) + t_2(0, 1, 1) \\&= (2 + t_1, 1 + t_2, 3 + t_1 + t_2).\end{aligned}$$

Example.

If writing $x = (x_1, x_2, x_3)$ as any point in \mathbb{R}^3 , then we have the following parametric form

$$x_1 = 2 + t_1,$$

$$x_2 = 1 + 2t_2,$$

$$x_3 = 3 + t_1 + t_2.$$

Example.

Find the vector and parametric equations of the plane in \mathbb{R}^4 that passes through the point $\mathbf{x}_0 = (2, -1, 0, 3)$ and is parallel to both $\mathbf{v}_1 = (1, 5, 2, -4)$ and $\mathbf{v}_2 = (0, 7, -8, 6)$.

Solution. The vectors \mathbf{v}_1 and \mathbf{v}_2 are not collinear. In the vector form,

$$\mathbf{x} = (x_1, x_2, x_3, x_4) = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

In the parametric form,

$$x_1 = 2 + t_1,$$

$$x_2 = -1 + 5t_1 + 7t_2,$$

$$x_3 = 2t_1 - 8t_2,$$

$$x_4 = 3 - 4t_1 + 6t_2.$$

Lines through two points in \mathbb{R}^2 .

Find vector and parametric equations for the line in \mathbb{R}^2 that passes through the points $P(0, 7)$ and $Q(5, 0)$.

Solution. The two points form a vector in \mathbb{R}^2 :

$$\mathbf{v} = (5, 0) - (0, 7) = (5, -7).$$

Thus in the vector form

$$\mathbf{x} = (0, 7) + t\mathbf{v} = (0, 7) + (5t, -7t) = (5t, 7 - 7t).$$

In the parametric form

$$x_1 = 5t,$$

$$x_2 = 7 - 7t.$$

Line Segment.

Definition. If \mathbf{x}_0 and \mathbf{x}_1 are vectors in \mathbb{R}^n , then the equation

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0), \quad (0 \leq t \leq 1)$$

defines **the line segment** from \mathbf{x}_0 to \mathbf{x}_1 . When convenient, this equation can be also written as

$$\mathbf{x} = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1, \quad 0 \leq t \leq 1.$$

Line Segment from one point to another point in \mathbb{R}^2 .

Find the equation of the line segment in \mathbb{R}^2 from $\mathbf{x}_0 = (1, -3)$ and $\mathbf{x}_1 = (5, 6)$.

Solution. The line segment, for $0 \leq t \leq 1$,

$$\begin{aligned}\mathbf{x} &= \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \\ &= (1, -3) + t(4, 9).\end{aligned}$$

or it can be written as

$$(1 - t)\mathbf{x}_0 + t\mathbf{x}_1 = (1 - t)(1, -3) + t(5, 6), \quad 0 \leq t \leq 1.$$

Dot product form of a linear system.

A linear equation in the variables x_1, x_2, \dots, x_n has the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b,$$

for the coefficients a_1, a_2, \dots, a_n not all zero. This equation can be written as

$$\mathbf{a} \cdot \mathbf{x} = b.$$

The corresponding homogeneous equation is

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0,$$

for the coefficients a_1, a_2, \dots, a_n not all zero. Similarly

$$\mathbf{a} \cdot \mathbf{x} = 0.$$

Dot product form of a linear system.

Consider the homogeneous system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = 0, \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = 0. \end{cases}$$

If we denote the successive row vectors of the coefficient matrix by $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$, then we can rewrite this system in dot product as

$$\begin{cases} \mathbf{r}_1 \cdot \mathbf{x} = 0, \\ \mathbf{r}_2 \cdot \mathbf{x} = 0, \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} = 0. \end{cases}$$

This shows that every solution vector \mathbf{x} is orthogonal to every row vector of the coefficients matrix.

Conclusion. If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system

$$Ax = \mathbf{0}$$

consists of all vectors in \mathbb{R}^n that are orthogonal to every row vector of A .

Orthogonality of row vectors and solution vectors.

Consider the homogeneous linear system

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0.$$

In the vector form,

$$\mathbf{x} = (-3r - 4s - 2t, r, -2s, s, t, 0).$$

This solution is orthogonal to every row vector in the matrix A .

The relationship between $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$.

Theorem. The general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.

Proof.

Proof. Let \mathbf{x}_0 be any specific solution of $A\mathbf{x} = \mathbf{b}$. Let W be the solution set of $A\mathbf{x} = \mathbf{0}$ and V be the solution set of $A\mathbf{x} = \mathbf{b}$. We first prove that $\mathbf{x}_0 + W \subset V$.

For any $w \in W$, we show that $\mathbf{x}_0 + w$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$:

$$A(\mathbf{x}_0 + w) = A\mathbf{x}_0 + Aw = \mathbf{0} + \mathbf{b} = \mathbf{b}.$$

Thus $\mathbf{x}_0 + w$ is a solution to the inhomogeneous linear system $A\mathbf{x} = \mathbf{b}$.

Secondly we prove that $\mathbf{V} \subset x_0 + \mathbf{W}$. On the other hand, let $y \in \mathbf{V}$. It is known that $x_0 \in \mathbf{V}$. Then

$$A(\mathbf{y} - \mathbf{x}_0) = A\mathbf{y} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

which shows that $\mathbf{y} - \mathbf{x}_0 \in \mathbf{W}$, i.e., $y \in x_0 + \mathbf{W}$.

To conclude the proof, we have

$$\mathbf{V} = x_0 + \mathbf{W}.$$

Homework and Reading.

Homework. Exercise. #2, #4, #6, #8, #10, #12, #15, #25. True or false questions on page 160.

Reading. Section 4.1.