Lecture 16: Section 4.1

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October 30, 2013

Real vector spaces

In this section, we discuss the definition of vector spaces. Then we give some examples which are vector spaces, and are not vector spaces.

Vector Space Axioms.

Definition. Let **V** be an arbitrary nonempty set of objects on which two operations are defined: **addition, and multiplication by scalars.**

By **addition**, we mean a rule for associating with each pair of objects **u** and **v** in **V** an object **u** + **v**, called the **sum** of **u** and **v**.

By **scalar multiplication**, we mean a rule for associating with each scalar k and each object **u** in **V** an object k**u**, called the scalar multiple of **u** by k. If the following axioms are satisfied by all objects **u**, **v**, **w** in **V** and all scalars k and m, then we call **V** a **vector space** and we call the objects in **V** vectors.

Addition part.

(1). If **u** and **v** are objects in **V**, then **u** + **v** are in **V**. (**Closed under addition.**)

(2). **u** + **v** = **v** + **u**.

(3). **u** + (**v** + **w**) = (**u** + **v**) + **w**.

(4). There is an object **0** in **V**, called a **zero vector** for **V**, such that **0** + **u** = **u** + **0** = **u** for all **u** in **V**.

(5). For each **u** in **V**, there is an object −**u** in **V**, called a **negative** of **u** such that **u** + (−**u**) = (−**u**) + **u** = **0**.

Scalar multiplication part.

(6). If k is any scalar and $\mathbf{u}$ is any object in $\mathbf{V}$, then $k\mathbf{u}$ is in $\mathbf{V}$. (**Closed under scalar multiplication.**)

(7). $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$.

(8). $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$.

(9). $k(m\mathbf{u}) = (km)\mathbf{u}$.

(10). $1\mathbf{u} = \mathbf{u}$.

Key points. "A nonempty set" + "Two operations on this set: addition and scalar multiplication."

Example 1.

Let $\mathbf{V} = \{\mathbf{0}\}$, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0},$$
$$k\mathbf{0} = \mathbf{0}$$

for all scalars k. Then we verify that $\mathbf{V}$ is a vector space. We call this vector space the **zero vector space.**

Proof of Example 1.

(1). $\mathbf{0} + \mathbf{0} = \mathbf{0}$ is in $\mathbf{V}$.

(2). $\mathbf{0} + \mathbf{0}$ is the same as $\mathbf{0} + \mathbf{0}$; both of them is $\mathbf{0}$.

(3). $\mathbf{0} + (\mathbf{0} + \mathbf{0}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$, which is the same as $(\mathbf{0} + \mathbf{0}) + \mathbf{0}$.

(4). $\mathbf{0}$ is the zero vector.

(5). For $\mathbf{0}$ in $\mathbf{V}$, there is $\mathbf{0}$ itself, such that $\mathbf{0} + \mathbf{0} = \mathbf{0}$.

(6). For k is any scalar and $\mathbf{0}$ is any object in $\mathbf{V}$, $k\mathbf{0} = \mathbf{0}$ is in $\mathbf{V}$.

(7). $k(\mathbf{0} + \mathbf{0}) = k\mathbf{0} = \mathbf{0}$, while $k\mathbf{0} + k\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Hence $k(\mathbf{0} + \mathbf{0}) = k\mathbf{0} + k\mathbf{0}$.

(8). $(k + m)\mathbf{0} = \mathbf{0}$, while $k\mathbf{0} + m\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Hence $(k + m)\mathbf{0} = k\mathbf{0} + m\mathbf{0}$.

(9). $k(m\mathbf{0}) = k\mathbf{0} = \mathbf{0}$ while $(km)\mathbf{0} = \mathbf{0}$.

(10). $1\mathbf{0} = \mathbf{0}$.

Example 2.

Let $V = \mathbb{R}^n$, and define the vector space operations on **V** to be the usual operations of addition and scalar multiplication of n-tuples; that is

$$
\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) \\
&= (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n), \\
k\mathbf{u} &= k(u_1, u_2, \cdots, u_n) \\
&= (ku_1, ku_2, \cdots, ku_n).
\end{aligned}
$$

V is a vector space.

Proof: Addition.

Firstly we see that these two operations are closed under addition and scalar multiplication.

(2). $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$;
$\mathbf{v} + \mathbf{u} = (v_1 + u_1, v_2 + u_2, \cdots, v_n + u_n)$. Since for each j, $u_j + v_j = v_j + u_j$, we have $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(3). $\mathbf{u} + (\mathbf{v} + \mathbf{w})$ is the same as $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$. This is proven similarly as in (2).

(4). There exists $\mathbf{0} = (0, 0, \cdots, 0)$, the zero vector.

(5). For each $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, there exists $-\mathbf{u} = (-u_1, -u_2, \cdots, -u_n)$ such that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

Proof. Scalar multiplication.

Secondly regarding scalar multiplications, we have

(7). $k(\mathbf{u} + \mathbf{v}) = (ku_1 + kv_1, ku_2 + kv_2, \cdots, ku_n + kv_n)$. On the the hand,

$$k\mathbf{u} + k\mathbf{v} = (ku_1, ku_2, \cdots, ku_n) + (kv_1, kv_2, \cdots, kv_n)$$
$$= (ku_1 + kv_1, ku_2 + kv_2, \cdots, ku_n + kv_n).$$

Hence

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}.$$

Similarly $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$, $k(m\mathbf{u}) = (km)(\mathbf{u})$, and $1\mathbf{u} = \mathbf{u}$.

So $\mathbb{R}^n$ is a vector space.

The vector space of infinite sequences of real numbers.

Let $\mathbf{V} = \{(u_1, u_2, \cdots, u_n) : u_i \in \mathbb{R}, 1 \le i \le n\}$. Then we define two operations:

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n, \cdots) \\
&= (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n, \cdots). \\
k\mathbf{u} &= (ku_1, ku_2, \cdots, ku_n, \cdots), \text{ where k is a scalar.}
\end{aligned}$$

Proof.

Solution. Firstly we see that these two operations are closed under addition and scalar multiplication.

Secondly the zero vector is $\mathbf{0} = (0, 0, \cdots, 0)$.

For each $\mathbf{u} \in \mathbf{V}$, the negative of $\mathbf{u}$ is

$$(-u_1, -u_2, \cdots, -u_n).$$

The rest rules are easily verified.

Example.

Let
$$\mathbf{V} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 2 \right\}.$$

Define two operations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix};$$

$$k \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}.$$

Proof.

Solution. Firstly we see that these two operations are closed under addition and scalar multiplication.

Secondly the zero vector is $\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$.

For each $\mathbf{u} \in \mathbf{V}$, the negative of $\mathbf{u}$ is

$$\left[\begin{array}{cc} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{array} \right].$$

The rest rules are easily verified. So V is a vector space. Similarly for any set of $m \times n$ matrix.

Example.

Let **V** be the set of real-valued functions that are defined at each x in the interval $(-\infty, \infty)$. Let f and g are two functions in **V** and if k is any scalar, we define two operations: addition and scalar multiplication.

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$$
$$k\mathbf{f}(x) = kf(x).$$

Proof.

Solution. Firstly we see that these two operations are closed under addition and scalar multiplication.

Secondly the zero vector is $f(x) \equiv 0$ for any $x \in (-\infty, \infty)$.

For each $\mathbf{f} \in \mathbf{V}$, the negative of $\mathbf{f}$ is $-\mathbf{f}(x) = -f(x)$ for any $x \in (-\infty, \infty)$. The rest rules are easily verified. Hence V is a vector space.

A set that is not a vector space.

Let $V = \mathbb{R}^2$. Define two operations

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2),$$
$$k\mathbf{u} = (ku_1, 0).$$

This is a set with two operations. But the rule

$$1\mathbf{u} = \mathbf{u}$$

fails because $1\mathbf{u} = (u_1, 0)$ which is not (u_1, u_2) if $u_2 \neq 0$.

Theorem 4.1.1. Let $\mathbf{V}$ be a vector space, $\mathbf{u}$ a vector in $\mathbf{V}$, and k is a scalar. Then

$$0\mathbf{u} = \mathbf{0},$$
$$k\mathbf{0} = \mathbf{0},$$
$$(-1)\mathbf{u} = -\mathbf{u},$$
$$\text{If } k\mathbf{u} = \mathbf{0}, \text{then } k = 0, \text{ or } \mathbf{u} = 0.$$

It is clear that the Theorem holds true for the usual vector space $\mathbb{R}^n$. Here is a general vector space, we prove it by following the rules defining the vector space.

Proof.
(a). We write $0 = 0 + 0$. Then

$$0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}.$$

The second follows from the distribution law. Then we add the negative of $0\mathbf{u}$ to both sides

$$0\mathbf{u} - 0\mathbf{u} = (0\mathbf{u} + 0\mathbf{u}) - 0\mathbf{u} = 0\mathbf{u} + (0\mathbf{u} - 0\mathbf{u}) = 0\mathbf{u} + \mathbf{0}.$$

That is to say,

$$\mathbf{0} = 0\mathbf{u}.$$

Similarly we prove that the second equality. $\qquad\square$

Cont.

We prove the third equality by using the first equality: since $0 = 1 - 1$,

$$0\mathbf{u} = (1 + (-1))\,\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u}.$$

By using the part **(a)** and the rule,

$$\mathbf{0} = \mathbf{u} + (-1)\mathbf{u}.$$

We add $-\mathbf{u}$ to both sides to obtain

$$-\mathbf{u} = (-1)\mathbf{u}.$$

For the last part: If $k \neq 0$, then by part **(b)**,

$$\frac{1}{k}\left(k\mathbf{u}\right) = \frac{1}{k}\mathbf{0} = \mathbf{0}.$$

Hence

$$(\frac{1}{k} \times k)\mathbf{u} = \mathbf{0}.$$

Thus

$$1\mathbf{u} = \mathbf{0}.$$

Thus $\mathbf{u} = \mathbf{0}$.

Homework and Reading.

Homework. Ex. #1, # 2,# 4, # 7,# 11,# 12,#18. True or false questions on page 179.

Reading. Section 4.2.