Lecture 17: Section 4.2

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Subspaces

We will discuss subspaces of vector spaces.

Subspaces

Definition. A subset W is a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Since W is a subset of V, certain axioms holding for V apply to vectors in W. For instance, if $\mathbf{v_1}, \mathbf{v_2} \in W$,

$$\mathbf{v_1} + \mathbf{v_2} = \mathbf{v_2} + \mathbf{v_1}.$$

So to say a subset W is a subspace of V, we need to verify that W is closed under addition and scalar multiplication.

Theorem. If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions hold.

(a). If $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$.

(b). If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.

Proof.

Proof. The zero vector is in W because we can take $k = 0$. Given $\mathbf{u} \in W$, $-\mathbf{u} \in W$. The rest axioms holding for V are also true for W. So W is a vector space. Hence W is a subspace of V.

Zero vector space.

Example. Let V be any vector space and $W = \{\mathbf{0}\}$. Then W is a subspace of V because

$$\mathbf{0} + \mathbf{0} = \mathbf{0}, \text{ and } k\mathbf{0} = \mathbf{0},$$

for any scalar k.

Example.

Lines through the origin are subspaces of $\mathbb{R}^2$ or $\mathbb{R}^3$. Let l be a line in $\mathbb{R}^2$ through the origin, denoted by W, then for any two vectors $\mathbf{v_1}, \mathbf{v_2} \in W$,

$$\mathbf{v_1} = t_1\mathbf{v},$$
$$\mathbf{v_2} = t_2\mathbf{v},$$

where $\mathbf{v}$ is the direction of the line l. Then

$$\mathbf{v_1} + \mathbf{v_2} = (t_1 + t_2)\mathbf{v};$$

so $\mathbf{v_1} + \mathbf{v_2}$ is on the line l. On the other hand, for any scalar k,

$$k\mathbf{v_1} = (kt_1)\mathbf{v}.$$

Hence $k\mathbf{v_1}$ is on the line l. So l is a subspace.

Note that lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ not through the origin are not subspaces because the origin $\mathbf{0}$ is not on the lines.

Example.

Planes through the origin are subspaces in $\mathbb{R}^3$. This can be proven similarly as in the previous example.

Example.

Let $V = \mathbb{R}^2$. Let $W = \{(x, y) : x \geq 0, y \geq 0\}$. This set is not a subspace of $\mathbb{R}^2$ because W is not closed under scalar multiplication. For instance, $\mathbf{v_1} = (1, 2) \in W$, but $-\mathbf{v_1} = (-1, -2)$ is not in the set W.

Subspaces of $M_{n\times n}$.

Let W be the set of symmetric matrices. Then W is a subspace of V because the sum of two symmetric matrices and the scalar multiplication of symmetric matrices are in W.

A subset of $M_{n\times n}$ is not a subspace.

Let W be the set of invertible $n \times n$ matrices. W is not a subspace because the zero matrix is not in W.

Note that to see whether a set is a subspace or not, one way is to see whether the zero vector is in the set or not.

The subspace $C(-\infty, \infty)$.

Let $\mathbf{V}$ be a vector space of functions on $\mathbb{R}$, and $W = C(-\infty, \infty)$, the set of continuous functions on $\mathbb{R}$. Then the sum of two continuous functions and scalar multiplication of continuous functions are still continuous functions. Then W is a subspace of $\mathbf{V}$.

The subspace of all polynomials.

Let $\mathbf{V}$ be a vector space of functions on $\mathbb{R}$, and W be the subset of all polynomials on $\mathbb{R}$. Then W is a subspace of $\mathbf{V}$.

Theorem.

Theorem. If $W_1, W_2, \cdots, W_r$ are subspaces of a vector space $\mathbf{V}$ and let W be the intersection of these subspaces, then W is also a subspace of $\mathbf{V}$.

Proof. Let $v_1, v_2 \in W$, then for any $1 \leq i \leq r$,

$$v_1, v_2 \in W_i.$$

Then

$$v_1 + v_2 \in W_i \text{ for any } i.$$

Hence $v_1 + v_2 \in W$. On the other hand, for any scalar, $k\mathbf{v_1} \in W_i$ for any i. Therefore $kv_1 \in W$. Thus W is a subspace of $\mathbf{V}$.

Remark. The union of the two subspaces V_1, V_2 of $\mathbf{V}$ is not a subspace of $\mathbf{V}$. For instance, let l_1 and l_2 be two lines through the origin in $\mathbb{R}^2$. We know that l_1, l_2 are subspace of $\mathbb{R}^2$. Take v_1, v_2 be two vectors on l_1, l_2. Thus by the parallelogram rule of sum of two vectors, $v_1 + v_2$ is not in $V_1 \cup V_2$. Thus $V_1 \cup V_2$ is not a subspace of $\mathbb{R}^2$.

Definition. If w is a vector in a vector space $\mathbf{V}$, then w is said to be a **linear combination** of the vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}$ in $\mathbf{V}$ if w can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_r\mathbf{v_r}$$

for some scalars $k_1, k_2, \cdots, k_r$. Then these scalars are called the coefficients of the linear combination.

Theorem. If $S = \{\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_r}\}$ is a nonempty set of vectors in a vector space **V**, then

(a). The set **W** of all possible linear combinations of the vectors in S is a subspace of **V**.

(b). The set **W** in part (a) is the "smallest" subspace of **V** that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains **W**.

Proof. **Part (a).** Let $\mathbf{u} = c_1\mathbf{w_1} + c_2\mathbf{w_2} + \cdots + c_r\mathbf{w}_r$ and $\mathbf{v} = k_1\mathbf{w_1} + k_2\mathbf{w_2} + \cdots + k_r\mathbf{w}_r$. Then

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{w_1} + (c_2 + k_2)\mathbf{w_2} + \cdots + (c_r + k_r)\mathbf{w}_r,$$
$$k\mathbf{u} = (k_1 c_1)\mathbf{w_1} + (k_2 c_2)\mathbf{w_2} + \cdots + (k_r c_r)\mathbf{w}_r.$$

Thus $\mathbf{W}$ is a subspace.

Part (b). Let $\mathbf{V_1}$ be a subspace of $\mathbf{V}$ and contains all the linear combinations of $\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_r}$. Then

$$W \subset \mathbf{V}_1.$$

Definition. The subspace of a vector space **V** that is formed from all possible linear combinations of the vectors in a nonempty set **S** is called the **span of S**, and we say that the vectors in S **span** that subspace.

If $S = \{\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_r}\}$, then we denote the span of S by $\text{span}(\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_r})$, $\text{span}(S)$.

Example.

Example. The standard unit vectors span $\mathbb{R}^n$. Recall that the standard unit vectors in $\mathbb{R}^n$ are

$$\mathbf{e}_1 = (1, 0, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, 0 \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \cdots, 0, 1).$$

Proof. Any vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is a linear combination of $\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}$ because

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

Thus

$$\mathbb{R}^n \subset \text{span}(\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}).$$

Hence

$$\mathbb{R}^n = \text{span}(\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}).$$

Example.

Let $\mathbf{P}_n$ be a set of all the linear combinations of polynomials $1, x, x^2, \cdots, x^n$. Thus

$$\mathbf{P}_n = \text{span}(1, x, x^2, \cdots, x^n).$$

Linear combinations.

Let $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\mathbb{R}^3$. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is not a linear combination of $\mathbf{u}, \mathbf{v}$.

Solution. Let $(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2)$. Thus

$$\begin{aligned} k_1 + 6k_2 &= 9, \\ 2k_1 + 4k_2 &= 2, \\ -k_1 + 2k_2 &= 7. \end{aligned}$$

Thus

$$k_1 = -3, \ k_2 = 2.$$

Cont.

Solution. Suppose that there exist k_1 and k_2 such that

$$\mathbf{w}' = (4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2).$$

Thus

$$\begin{aligned} k_1 + 6k_2 &= 4, \\ 2k_1 + 4k_2 &= -1, \\ -k_1 + 2k_2 &= 8. \end{aligned}$$

Therefore from the first and third equations, we have

$$k_1 = -5,\ k_2 = \frac{3}{2}.$$

But this solution does not satisfy the second equation.

Testing for spanning.

Determine whether $\mathbf{v_1} = (1, 1, 2)$, $\mathbf{v_2} = (1, 0, 1)$ and $\mathbf{v_3} = (2, 1, 3)$ span the vector space $\mathbb{R}^3$.

Solution. We know that

$$\text{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) \subset \mathbb{R}^3.$$

For any vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, there exists k_1, k_2 and k_3 such that

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3.$$

Thus

$$\begin{cases} k_1 + k_2 + 2k_3 & = b_1, \\ k_1 \qquad + k_3 & = b_2, \\ 2k_1 + k_2 + 3k_3 & = b_3. \end{cases}$$

The coefficient matrix is in form of

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

The determinant of the coefficient matrix is

$$- \begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix} - \begin{matrix} 1 & 2 \\ 1 & 1 \end{matrix} = 2 \neq 0.$$

Theorem. The solution set of a homogeneous linear system $\mathbf{A}x = \mathbf{0}$ in n unknowns is subspace of $\mathbb{R}^n$.

Solution. Let x_1, x_2 be two solutions to the linear system $\mathbf{A}x = \mathbf{0}$. Then

$$A(x_1 + x_2) = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

and

$$A(kx_1) = kA(x_1) = k\mathbf{0} = \mathbf{0}.$$

Thus the solution set is a subspace of $\mathbb{R}^n$.

Homework and Reading.

Homework. Ex. # 1, # 2, # 4, # 5, # 7, # 8, # 14, # 15.
True or false questions on page 190.

Reading. Section 4.3.