Lecture 18: Section 4.3

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Linear Independence and Linear Dependence.

We will discuss linear independence of vectors in a vector space.

Definition. If $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ is a nonempty set of vectors in a vector space $\mathbf{V}$, then the vector equation

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_r\mathbf{v_r} = 0$$

has at least one solution, namely,

$$k_1 = 0, k_2 = 0, \cdots, k_r = 0.$$

We call it the **trivial solution.** If this is the only solution, then S is said to be a **linearly independent set**. If there are solutions in addition to the trivial solution, then $\mathbf{S}$ is said to be a linearly dependently set.

Example 1.

Example 1. Linear independence of the standard unit vectors in $\mathbb{R}^n$. Let

$$e_1 = (1, 0, 0 \cdots, 0), e_2 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 1).$$

Let

$$c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = \mathbf{0}.$$

Thus

$$(c_1, c_2, \cdots, c_n) = (0, 0, \cdots, 0).$$

This implies that $c_1 = 0, c_2 = 0, \cdots, c_n = 0$. Therefore $\{e_1, e_2, \cdots, e_n\}$ is linearly independently.

Example 2.

Example 2. Linear independence in $\mathbb{R}^3$. Determine whether the vectors $\mathbf{v_1} = (1, -2, 3), \mathbf{v_2} = (5, 6, -1)$ and $\mathbf{v_3} = (3, 2, 1)$ are linearly independent or linearly dependent in $\mathbb{R}^3$.

Solution. Suppose that there exist k_1, k_2 and k_3 such that

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + k_3\mathbf{v_3} = (0, 0, 0).$$

Thus

$$(k_1 + 5k_2 + 3k_3, -2k_1 + 6k_2 + 2k_3, 3k_1 - k_2 + k_3) = (0, 0, 0). \quad (1)$$

Writing it in the linear system,

$$\begin{cases} k_1 + 5k_2 + 3k_3 & = 0, \\ -2k_1 + 6k_2 + 2k_3 & = 0, \\ 3k_1 - k_2 + k_3 & = 0. \end{cases}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix}.$$

By elementary row reductions,

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last row only consists of zero, there exists nontrivial solutions (k_1, k_2, k_3) satisfying the equation (1).

Linear independence in $\mathbb{R}^4$.

Determine whether the vectors

$$\mathbf{v_1} = (1, 2, 2, -1), \mathbf{v}_2 = (4, 9, 9, -4), \mathbf{v}_3 = (5, 8, 9, -5)$$

in $\mathbb{R}^4$ are linearly dependent or linearly independent.

Proof. Suppose that there exists k_1, k_2, k_3 such that

$$k_1\mathbf{v_1} + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = (0, 0, 0).$$

Suppose that there exist k_1, k_2, k_3 such that

$$\begin{cases} k_1 + 4k_2 + 5k_3 & = 0, \\ 2k_1 + 9k_2 + 8k_3 & = 0, \\ 2k_1 + 9k_2 + 9k_3 & = 0, \\ -k_1 - 4k_2 - 5k_3 & = 0. \end{cases}$$

The 4th equation is the same as the first equation. So the augmented matrix is

$$\begin{bmatrix} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \end{bmatrix}$$

By the elementary row reductions, we have

$$\begin{bmatrix} 1 & 4 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

This implies that

$$k_1 = 0,\ k_2 = 0,\ k_3 = 0.$$

Therefore $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ is linearly independent.

An important linearly independently set in $\mathbb{P}_n$.

Show that the polynomials $1, x, \cdots, x^n$ form a linearly independent set in $\mathbb{P}_n$.

Solution. Suppose that there exists $a_0, a_1, \cdots, a_n$ such that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0.$$

Thus

$$a_0 = a_1 = a_2 = \cdots = a_n = 0.$$

Hence the set of polynomials of $\{1, x, \cdots, x^n\}$ is linearly independent.

Theorem.

Theorem. A set S with two or more vectors is

(a). Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

(b). Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.

Proof. We observe that the second statement follows from the first statement. So we only need to prove part (a).
Let $S = \{v_1, v_2, \cdots, v_k\}$.

"⇒". If $v_1 = c_2 v_2 + \cdots + c_k v_k$, then

$$0 = v_1 - c_2 v_2 + \cdots - c_k v_k.$$

Thus there exists nonzero solution $(1, -c_2, \cdots, -c_k)$ satisfying the equation.

" $\Leftarrow$ ". Suppose that there exists S is linearly dependent, i.e., there exists $(c_1, c_2, \cdots, c_k)$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0.$$

Suppose that $c_1 \neq 0$,

$$v_1 = -\frac{c_2}{c_1} v_2 - \cdots - \frac{c_k}{c_2} v_k.$$

This proves the statement.

Theorem.

(a). A finite set that contains $bf0$ is linearly dependent.

(b). A set with exactly one vector is linearly dependent if and only if that vector is nonzero.

(c). A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

The proof is omitted.

Theorem. Let $S = \{v_1, v_2, \cdots, v_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

Solution. Suppose that there exists $x_1, x_2, \cdots, x_r$ such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_r v_r = 0. \tag{2}$$

We need to show that there exists a nonzero solution $(x_1, x_2, \cdots, x_r)$ to this equation. Suppose that

$$v_i = (a_{1i}, a_{2i}, \cdots, a_{ni}).$$

Then we have the system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1r}x_r &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2r}x_r &= 0, \\ \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nr}x_r &= 0. \end{cases}$$

Then the augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} & 0 \\ a_{21} & a_{22} & \cdots & a_{2r} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} & 0. \end{bmatrix}$$

By Theorem 1.2.2, since this system has more unknowns than equations, it has infinitely many solutions. So there exists a nonzero solution to the equation (2).

By using the standard unit basis in $\mathbb{R}^n$, each v_i can be expressed as a linear combination of $e_1, e_2, \cdots, e_n$:

$$\begin{cases} v_1 & = a_{11}e_1 + a_{12}e_2 + \cdots + a_{1n}e_n \\ v_2 & = a_{21}e_1 + a_{22}e_2 + \cdots + a_{2n}e_n \\ \vdots & \vdots \\ v_n & = a_{n1}e_1 + a_{n2}e_2 + \cdots + a_{nn}e_n \\ \vdots & \vdots \\ v_r & = a_{r1}e_1 + a_{r2}e_2 + \cdots + a_{rn}e_n. \end{cases}$$

We write the first n equations in the following form:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Let A be the coefficient matrix.

We divide it into two cases.

Case 1. If A is invertible, then the column vector of $e_1, e_2, \cdots, e_n$ can be written as the product of the inverse matrix A and the column vector $v_1, v_2, \cdots, v_n$. Since v_r is a linear combination of $e_1, e_2, \cdots, e_n$. Thus v_r is a linear combination of $v_1, v_2, \cdots, v_n$. Hence the set of vectors $\{v_1, v_2, \cdots, v_r\}$ is linearly dependent.

Case 2. If A is not invertible, then by elementary row operations, we can reduce the last row of the matrix A to the zero row. Applying the same elementary row operations to the left hand side, we see that a linear combination of $\{v_1, v_2, \cdots, v_n\}$ with nonzero coefficients is equal to zero. Thus $\{v_1, \cdots, v_n\}$ is linearly dependent. Hence $\{v_1, \cdots, v_n, \cdots, v_r\}$ is linearly dependent.

Homework and Reading.

Homework. Ex. # 2, # 3, # 4, # 6, # 10, # 12, # 13, # 16.
True or false questions on page 200.

Reading. Section 4.4.