Lecture 19: Section 4.4

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Coordinate System in Linear Algebra.

(1). Recall that $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ is linearly independent if the equation

$$c_1\mathbf{v_1} + \cdots + c_r\mathbf{v_r} = 0$$

implies that

$$c_1 = c_2 = \cdots = c_r = 0.$$

(2). Recall that **the span of the set** $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ in a vector space $\mathbf{V}$: span(S) is the collection of all linear combinations of $\mathbf{v_1}, \cdots, \mathbf{v_r}$.

Definition. If bfV is any vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ is a finite set of vectors in $\mathbf{V}$, then S is called a basis for $\mathbf{V}$ if the following two conditions hold:

(a). S is linearly independent.

(b). S spans $\mathbf{V}$.

Example 1. The standard basis for $\mathbb{R}^n$.

The standard unit vectors

$$\mathbf{e_1} = (1, 0, \cdots, 0), \mathbf{e_2} = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_n = (0, 0, \cdots, 1)$$

spans $\mathbb{R}^n$ because

(a). The vectors $\mathbf{e_1}, \cdots, \mathbf{e_n}$ are linearly independent.

$$c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n = (c_1, \cdots, c_n) = 0$$

implies that

$$c_1 = c_2 = \cdots = c_n = 0.$$

(b). The set S spans $\mathbf{V}$ because

$$(x_1, \cdots, x_n) = x_1 e_1 + \cdots + x_n e_n.$$

The standard basis for $\mathbb{P}_n$.

Show that $S = \{1, x, \cdots, x^n\}$ is a basis for the vector space $\mathbb{P}_n$ of polynomials of degree n or less.

Proof.

(a). The set S is linearly independent because

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$$

implies that $c_0 = c_1 = \cdots = c_n = 0$.

(b). The set S spans $\mathbf{P}_n$.

□

Another basis for $\mathbb{R}^3$.

Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$ and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

Proof.

(a). We show that it is linearly independent. Suppose that there exist c_1, c_2, c_3 satisfying

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0.$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

□

By an elementary row reduction,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -3 \\ 0 & -2 & 1 \end{bmatrix}.$$

Thus the determinant is $1 \times (5 - 6) = -1$. Since the matrix A is invertible, $c_1 = c_2 = c_3 = 0$.

(b). We show that S spans $\mathbb{R}^3$. For any vector (b_1, b_2, b_3), we prove that there exists c_1, c_2, c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (b_1, b_2, b_3).$$

The coefficient matrix is still A and by part (a), it is invertible. So there exists a solution (c_1, c_2, c_3).

Example 3. The standard basis for M_{mn}.

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

Remark. These are called the standard basis for M_{22}.

A vector space that has no finite spanning set.

Show that the vector space of P_∞ of all polynomials with real coefficients has no finite spanning set.

Proof. Recall that P_∞ is the collection of all polynomials. Show that there exists a finite spanning set $P = \{\mathbf{p_1}, \mathbf{p_2}, \cdots, \mathbf{p_n}\}$ for P_∞. We have n numbers

$$\deg(p_1), \cdots, \deg(p_n).$$

We denote the maximum one is d, which will lead to a contradiction. Indeed, any polynomial of degree $d+1$ can not be expressible as a linear combination of $\{\mathbf{p_1}, \mathbf{p_2}, \cdots, \mathbf{p_n}\}$.

Some finite and infinite dimensional spaces.

finite dimensional space. $\mathbb{R}^n$, P_n and M_{mn} are of finite dimensional spaces.

infinite dimensional space. P_∞ are of infinite dimensional. Another example is C^∞.

Coordinates relative to a basis.

Theorem. If $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for a vector space $\mathbf{V}$, then every vector $\mathbf{v}$ in $\mathbf{V}$ can be expressed in form

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

in exactly one way.

Proof.

Suppose that there exist two ways of representation, i.e., there exist $(c_1, \cdots, c_r)$ and $(d_1, \cdots, d_r)$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n.$$

Then

$$(c_1 - d_1)\mathbf{v}_1 + \cdots + (c_n - d_n)\mathbf{v}_n = 0.$$

Since $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis, S is linearly independent. Thus

$$c_1 = d_1, c_2 = d_2, \cdots, c_n = d_n.$$

Coordinate matrix, Coordinate vector.

Definition. If $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for a vector space $\mathbf{V}$, and

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

is the expression for a vector space $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, \cdots, c_n$ are called the coordinates of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, \cdots, c_n)$ in $\mathbb{R}^n$ constructed from these coordinates is called the coordinate vector of $\mathbf{v}$ relative to S; it is denoted by

$$(\mathbf{v})_\mathbf{S} = (\mathbf{c_1}, \mathbf{c_2}, \cdots, \mathbf{c_n}).$$

Coordinates relative to the standard basis for $\mathbb{R}^n$.

For $\mathbf{V} = \mathbb{R}^n$ and S is the standard basis, the coordinate vector $(\mathbf{v})_S$ and the vector $\mathbf{v}$ are the same,

$$\mathbf{v} = \mathbf{v}_S.$$

For example, take $n = 3$, the representation of a vector $\mathbf{v} = (a, b, c)$ as a linear combination of the vectors in the standard basis $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Coordinate vectors relative to standard bases.

(a). Find the coordinate vector for the polynomial

$$\mathbf{p}(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

relative to the standard basis for the vector space P_n.

(b). Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22}.

Proof.

The standard basis for P_n is $\{1, x, \cdots, x^n\}$. So the coordinate vector is

$$(c_0, c_1, \cdots, c_n).$$

Since the matrix B can be expressible as a linear combination of the standard basis for M_{22}:

$$B = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

So the coordinate vector is

$$(a, b, c, d).$$

□

Coordinates in $\mathbb{R}^3$.

(a). We showed that the vectors

$$\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 0, 0), \mathbf{v}_3 = (3, 3, 4)$$

form a basis for $\mathbb{R}^3$. Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b). Find the vector $\mathbf{v}$ in $\mathbb{R}^3$ relative to S is $\mathbf{v}_S = (-1, 3, 2)$.

Proof.

To find $\mathbf{v}_S$, suppose that there exists (c_1, c_2, c_3) such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

In terms of the linear system,

$$\begin{cases} c_1 + 2c_2 + 3c_3 & = 5, \\ 2c_1 + 9c_2 + 3c_3 & = -1, \\ c_1 + 4c_3 & = 9. \end{cases}$$

Then in terms of matrix representation,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}.$$

We have shown that the matrix A is invertible. Therefore by solving the linear system, there exists a solution (c_1, c_2, c_3). □

Under the definition of $(\mathbf{v}_S)$, we obtain

$$\mathbf{v} = (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 = (11, 31, 7).$$

Homework and Reading.

Homework. Ex. #2,#4, #5, # 6, # 9, # 10, # 11, # 12. True or false questions on page 208.

Reading. Section 4.5.