Lecture 2 & 3

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1. Echelon forms of matrices
2. Methods of the Gauss-Jordan elimination and Gauss elimination.
3. Homogeneous linear system.

Elementary row operations.

Recall the following three elementary row operations.

1. Multiplying a row through by a nonzero constant.

2. Interchange two rows.

3. Add a constant times one row to another.

Reduced row echelon form and row echelon form

In previous lecture, we have seen the following matrix after a sequence of elementary row operations:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}. \tag{1}$$

and

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}. \tag{2}$$

The matrix (1) is in **reduced row echelon form**, and the matrix (2) is in **row echelon form.**

Definitions.

Definition of "reduced row echelon form."

1. If a row does not consist of entirely zeros, then the first nonzero number in the row is a 1. We call this number a leading 1.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist of entirely of zeros, then leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

The matrix (1) is in this form. If only the first three items are satisfied, the matrix is then in the row echelon form; for instance, (2).

Examples.

Reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{3}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

and

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

Examples.

Row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}. \tag{6}$$

and

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

and

$$\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

Augmented matrix in solving linear systems

Consider the augmented matrix in the reduced row echelon form for a linear system of 4 variables, x_1, x_2, x_3 and x_4:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}. \tag{9}$$

It can be written a linear system of equations

$$\begin{cases} x_1 & = 3, \\ x_2 & = -1, \\ x_3 & = 0, \\ x_4 & = 5. \end{cases}$$

This also gives the unique solution, $(x_1, x_2, x_3, x_4) = (3, -1, 0, 5)$. This is the only solution to the linear system.

Linear system in three unknowns.

Solve the following three linear systems of equations in three unknowns.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

and

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

and

$$\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Solving (10)

For the linear system (10), from the last row, we have

$$0x + 0y + 0z = 1,$$

i.e.,

$$0 = 1.$$

Hence the system is inconsistent. No solutions to the system.

Solving (11)

For the linear system (11), the last row has no effect on solutions.

$$0x + 0y + 0z = 0.$$

The linear system corresponds to the augmented matrix

$$\begin{cases} x + 3z & = -1, \\ y - 4z & = 2. \end{cases}$$

Let $z = t$, then

$$x = -1 - 3t,\ y = 2 + 4t.$$

Thus the solution is in the parameter form

$$(x, y, z) = (-1 - 3t, 2 + 4t, t),$$

where t is a parameter.

Solving (12)

For the linear system (12), the last two rows do not have effect on solutions. From the row, we write the corresponding equation

$$x - 5y + z = 4.$$

We write it in the parameter form

$$y = s,\ z = t,\ x = 4 + 5s - t.$$

Thus the solution is

$$(x, y, z) = (s, t, 4 + 5s - t),$$

where s, t are parameters.

The Gauss-Jordan elimination: Example 1.

We use an example to show how we apply the Gauss-Jordan elimination to solve a linear system.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}. \tag{13}$$

Solving (13)

Step 1. Exchange the first and the second row.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}.$$

Step 2. Multiply the first row by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}.$$

Step 3. Multiply -2 to the first row and add it to the third row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}.$$

Cont.

Step 4. Multiply the second row by $-\frac{1}{2}$.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & 6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}.$$

Step 5. Multiply the second row by -5 and add it to the third row

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}.$$

Cont.

Step 6. Multiply the third row by 2.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

If we stop at this step, the resulting matrix is in the row echelon form. The steps we perform so far are called "**Gaussian Elimination.**"
Now we begin with the last nonzero row and work upward. **Step 7.** Multiply the third row by $\frac{7}{2}$ and -6, and add them to the second row and the third row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Cont.

Step 8. Multiply the second row by 5, and add them to the second row and the third row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

The matrix is in the reduced row echelon form. All the steps leading to this form is called, **the Gauss–Jordan Elimination.**

The Gauss-Jordan elimination: Example 2.

Solve the linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 & = 0, \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 & = -1, \\ 5x_3 + 10x_4 + 15x_6 & = 5, \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6. \end{cases}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}.$$

Step 1. Add -2 times the first row to the second and the fourth row.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}.$$

Step 2. Multiply the second row by -1.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}.$$

Cont.

Step 3. Add -5 times the second row to the third row, and add -4 times the second row to the fourth row.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}.$$

Step 4. Interchange the last two rows.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Cont.

Step 5. Multiply $\frac{1}{6}$ to the third row.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 6. Add -3 times the third row to the second, and add 2 times the second row to the first row.

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding linear system is

$$\begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 & = 0, \\ x_3 + 2x_4 & = 0, \\ x_6 & = \frac{1}{3}. \end{cases}$$

Hence we have

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5, \\ x_3 &= -2x_4, \\ x_6 &= \frac{1}{3}. \end{aligned}$$

Choosing parameters $x_2 = r, x_4 = s$, and $x_5 = t$, we see that

$$x_1 = -3r - 4s - 2t, x_2 = r, x_3 = -2s, x_4 = s, x_5 = t, x_6 = \frac{1}{3}.$$

where r, s, t are parameters.

Definition. A system of linear equations is said to be homogeneous if the constant terms are all zero:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = 0, \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots & = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = 0. \end{cases}$$

The ordered n-tuple $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ is a solution. This is the trivial solution to this system. Other solutions are called nontrivial solutions.

Examples.

Example 3.

$$\begin{cases} a_1x + b_1y & = 0, \\ a_2x + b_2y & = 0. \end{cases}$$

Example 4.

$$\begin{cases} x_1 + 3x_2 - 2x_3 + \qquad 2x_5 & = 0, \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 & = 0, \\ \qquad 5x_3 + 10x_4 + 15x_6 & = 0, \\ 2x_1 + 6x_2 + \qquad 8x_4 + 4x_5 + 18x_6 & = 0. \end{cases}$$

Example.

Use the Gauss-Jordan elimination method to solve the homogeneous system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + \quad 2x_5 = 0, \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0, \\ 5x_3 + 10x_4 + 15x_6 = 0, \\ 2x_1 + 6x_2 + \quad 8x_4 + 4x_5 + 18x_6 = 0. \end{cases}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

Except for the last column, the augmented matrix is the same as that in **Example 2**. Observe that the elementary row operations do not change the zero column. So the reduced row echelon form is

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the corresponding linear system of equations is

$$\begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 & = 0, \\ x_3 + 2x_4 & = 0, \\ x_6 & = 0. \end{cases}$$

Choosing parameters $x_2 = r, x_4 = s$, and $x_5 = t$, we see that

$$x_1 = -3r - 4s - 2t, x_2 = r, x_3 = -2s, x_4 = s, x_5 = t, x_6 = 0.$$

where r, s, t are parameters.

Free variables in homogeneous linear systems

From the previous example, there are three parameters, $x_2 = r, x_4 = s$, and $x_5 = t,$. These are called free variables. x_1, x_3, x_6 can be expressed in term of the free variables, and are called the leading variables.

Theorem. If a homogeneous linear system has n unknowns , and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem. A homogeneous linear system with more unknowns than equations has infinitely many solutions.

Examples.

Suppose that the matrices below are augmented matrices for linear systems in the unknowns x_1, x_2, x_3 and x_4. Discuss the existence and uniqueness of solutions to the corresponding linear systems.

Example.

$$\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

This system is inconsistent because the last row gives that $0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$, i.e., $0 = 1$.

Examples Cont.

Example .

$$\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

The corresponding linear system is

$$\begin{cases} x_1 - 3x_2 + 7x_3 + 2x_4 & = 5, \\ x_2 + 2x_3 - 4x_4 & = 1, \\ x_3 + 6x_4 & = 9. \end{cases}$$

Setting $x_4 = t$, solving it backwards

$$x_3 = 9 - 6t,$$
$$x_2 = -17 + 16t,$$
$$x_1 = 88t - 109,$$

where t is a parameter.

Example.

$$\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{16}$$

The same matrix as before except for the last row. The last row implies that $x_4 = 0$. So let $t = 0$ in the previous solution, we get

$$x_1 = -109, x_2 = -17, x_3 = 9.$$

So this system has exactly one solution $(x_1, x_2, x_3, x_4) = (-109, -17, 9, 0)$.

Homework and Reading.

Homework. Ex. #2,#4, #6, #7, #12, #22, #30, # 41, and the True-False exercise on page 25.

Reading. Section 1.3.