Lecture 20: Section 4.5

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Dimension

In last section, we consider the basis for a vector space. In this section we take about the dimension of a vector space, which is related to the basis.

Theorem. All bases for a finite dimensional vector space have the same number of vectors.

Proof.

Suppose that there exist two basis for a vector space: $\{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ and $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$. We prove that $r = n$ by proving $r \leq n$ and $n \leq r$.

Step 1. Suppose that $r > n$. Since each vector in $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ can be expressible as a linear combination of $\{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$, there exist coordinate vectors

$$\mathbf{v}_i = a_{i1}\mathbf{e}_1 + a_{i2}\mathbf{e}_2 + \cdots + a_{in}\mathbf{e}_n.$$

□

We write it in the matrix form,

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_r \end{bmatrix} = A \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \end{bmatrix},$$

where A_{rn} is a matrix of $r \times n$. We apply the elementary row operations on A_{rn}; since $r > n$, at least the last row consists of zero. Applying the same elementary row operations to the left hand side. Since the last row of A_{rn} is zero, we see that there exists nonzero scalars $c_1, c_2, \cdots, c_r$ such that

$$c_1\mathbf{v_1} + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r = \mathbf{0}.$$

It implies that $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ is linearly dependent. A contradiction. So $r \leq n$. Similarly $n \leq r$. Thus $r = n$.

Definition. The **dimension** of a finite-dimensional vector space **V** is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for **V**. In addition, the zero vector space is defined to be have dimension zero.

Remark. By the previous theorem, any basis results in the same dimension number.

Dimensions of some familiar vector spaces.

Example.

- $\dim(\mathbb{R}^n) = n$.
- $\dim(P_n) = n + 1$.
- $\dim(M_{mn}) = mn$.

Dimension of Span(S).

If $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ is a linearly independent set in a vector space $\mathbf{V}$, then S is a basis for span(S) because

- S is linearly independent.
- S spans span(S).

We denote dim(span(S)) $= r$.

Dimension of a solution space.

Find the basis for and the dimension of the solution space of the homogeneous system

$$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 & = 0, \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 & = 0, \\ x_1 + x_2 - 2x_3 - x_5 & = 0, \\ x_3 + x_4 + x_5 & = 0. \end{cases}$$

Solution. The solution can be expressible as

$$x_1 = -s - t,\ x_2 = s,\ x_3 = -t,\ x_4 = 0,\ x_5 = t$$

which can be written in vector form as

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (-s - t, s, -t, 0, t) \\ &= s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1). \end{aligned}$$

The vectors $(-1,1,0,0,0)$ and $(-1,0,-1,0,1)$ are linearly independent because the equation

$$c_1(-1,1,0,0,0)+c_2(-1,0,-1,0,1)=\mathbf{0}$$

implies that

$$c_1=0,\ c_2=0.$$

On the other hand, these two vectors span the solution space. So the dimension of the solution space is 2.

Dimension of a solution space.

Find the basis for and the dimension of the solution space of the homogeneous system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 & = 0, \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 & = 0, \\ 5x_3 + 10x_4 + 15x_6 & = 0, \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 & = 0. \end{cases}$$

Solution. The solution can be expressible as

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t, x_6 = 0.$$

which can be written in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$$
$$= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, 2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0).$$

The vectors $(-3, 1, 0, 0, 0, 0)$, $(-4, 0, 2, 1, 0, 0)$ and $(-2, 0, 0, 0, 1, 0)$ are linearly independent because the equation

$$c_1(-3, 1, 0, 0, 0, 0) + c_2(-4, 0, 2, 1, 0, 0) + c_3(-2, 0, 0, 0, 1, 0) = \mathbf{0}$$

implies that

$$c_1 = 0,\ c_2 = 0,\ c_3 = 0.$$

On the other hand, these two vectors span the solution space. So the dimension of the solution space is 2.

Plus/Minus Theorem.

Theorem. Let S be a nonempty set of vectors in a vector space $\mathbf{V}$.
(a). If S is a linearly independent set, and if $\mathbf{v}$ is a vector in $\mathbf{V}$ that is outside of $\text{span}(S)$, then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.

(b). If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space, that is

$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\}).$$

Proof.

(a). Suppose that $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ denotes the vectors taken from $S \cup \{\mathbf{v}\}$. If these vectors are all taken S, they are linearly independent. If one of them is $\mathbf{v}$, let $\mathbf{v}_1 = \mathbf{v}$. We set the equation

$$c_1\mathbf{v} + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r = 0$$

If $c_1 = 0$, then $\{\mathbf{v_2}, \cdots, \mathbf{v_r}\}$ in S implies that

$$c_2 = \cdots = c_r = 0.$$

If $c_1 \neq 0$, then

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 - \cdots - \frac{c_r}{c_1}\mathbf{v}_r.$$

Thus $\mathbf{v}_1 \in \text{span}(S)$. A contradiction. So part (a) is proven. $\square$

(b). Firstly $S - \{\mathbf{v}\} \subset S$, then

$$\text{span}(S - \{\mathbf{v}\}) \subset \text{span}(S).$$

Secondly $\mathbf{v}$ can be expressible as a linear combination as other vectors in S,

$$\mathbf{v} \in \text{span}(S - \{\mathbf{v}\}).$$

Because of this, any linear combination in S involving $\mathbf{v}$ can be expressible as a linear combination of other vectors in S-$\{\mathbf{v}\}$. Therefore

$$\text{span}(S) \subset \text{span}(S - \{\mathbf{v}\}).$$

Thus

$$\text{span}(S - \{\mathbf{v}\}) = \text{span}(S).$$

Example 5. Show that $\mathbf{p}_1 = 1 - x^2$, $\mathbf{p}_2 = 2 - x^2$ and $\mathbf{p}_3 = x^3$ are linearly independent vectors.

Proof.

The set $S = \{\mathbf{p_1}, \mathbf{p_2}\}$ is linearly independent since neither vector in S is a scalar multiple of the other. Since the vector $\mathbf{p}_3$ can not be expressible as a linear combination of the vectors in S since x^3 is of degree 3, the set $\{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}\}$ is a linearly independently set. □

Theorem. Let $\mathbf{V}$ be an n-dimensional vector space, and let S be a set in $\mathbf{V}$ with exactly n vectors. Then S is a basis for $\mathbf{V}$ if and only if S spans $\mathbf{V}$ or S is linearly independent.

Proof.

One direction of implication is clear: if S is a basis, then S spans $\mathbf{V}$ or S is linearly independent.

On the other hand, let us suppose that $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ spans $\mathbf{V}$, and $\{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ is a basis of $\mathbf{V}$. We need to prove that S is linearly independent. We prove it by contradiction. Suppose that S is linearly dependent. Then by Theorem in the previous section, one vector is linear combination of other vectors in S; let this vector be $\mathbf{v}_n$. Then we have

$$\text{span}\, S = \text{span}(S - \{\mathbf{v}_n\}) = \text{span}(\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_{n-1}}\}).$$

For $1 \leq i \leq n-1$, each vector e_i can be expressible as a linear combination of vectors in S, $e_i = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n$. □

Writing it in the matrix form,

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = A_{n\times(n-1)} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix}.$$

Applying the elementary row operations to reduce $A_{n\times(n-1)}$ to the reduced row-echelon form; the last row only consists of zero. Apply the same row operations to the left hand side, we see that a nonzero linear combination of the row vectors is zero on the last row. This leads to a contradiction that $\{e_1, \cdots, e_n\}$ is linearly independent. So S is a basis.

Let us suppose that $\{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ is linearly independent. By using the same reasoning as above, there exists a matrix A such that

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = A_{n\times n} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}.$$

This matrix A is invertible. Apply the inverse, we see that $\{v_1, \cdots, v_n\}$ is a linear combination of $\{\mathbf{e_1}, \cdots, \mathbf{e_n}\}$. Since $\{\mathbf{e_1}, \cdots, \mathbf{e_n}\}$ spans $\mathbf{V}$, $\{v_1, \cdots, v_n\}$ spans $\mathbf{V}$. Thus S is a basis.

Bases by inspection.

(a). Explain why $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for $\mathbb{R}^2$. It is because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

(b). Explain why $\mathbf{v_1} = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$ and $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for $\mathbb{R}^3$. It is because these three vectors are linearly independent in $\mathbb{R}^3$. This way to prove this to consider that the equation that $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ implies that

$$c_1 = c_2 = \cdots = c_n = 0.$$

Theorem. Let S be a finite set of vectors in a finite dimensional vector space $\mathbf{V}$.

(a). If S spans $\mathbf{V}$ but is not a basis for $\mathbf{V}$, then S can be reduced to a basis for $\mathbf{V}$ by removing appropriate vectors from S.

(b). If S is a linearly independent set that is not already a basis for $\mathbf{V}$, then S can be enlarged to a basis for $\mathbf{V}$ by inserting appropriate vectors into S.

Proof.

Part (a). Suppose that $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ spans $\mathbf{V}$. Then by Theorem 4.5.2 (b), $r \geq n$. If $r = n$, by Theorem 4.5.4, S is a basis. Otherwise if $r > n$, we write

$$\begin{aligned} \mathbf{v}_1 &= c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + \cdots + c_{1r}\mathbf{v}_r, \\ \mathbf{v}_2 &= c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{2r}\mathbf{v}_r, \\ \vdots & \qquad\qquad \vdots \\ \mathbf{v}_r &= c_{r1}\mathbf{v}_1 + c_{r2}\mathbf{v}_2 + \cdots + c_{rr}\mathbf{v}_r. \end{aligned}$$

If $c_{ii} \neq 1$ for $1 \leq i \leq r$, then $\mathbf{v}_i$ can be expressible as a linear combination of other vectors in S; we remove this vector from S and the remaining set still spans $\mathbf{V}$. Suppose that we have removed m vectors, the rest $r - m$ vectors span $\mathbf{V}$ and $m \geq 1$ and $r - m \geq n$. We continue this process to arrive $r - m = n$. Therefore part (a) is proven. □

Part (b). Suppose that $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ is linearly independent; then by Theorem 4.5.2 (a), $r \leq n$. If $r = n$, then the set S is a basis because it contains exactly n vectors. On the other hand, let $\{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ be a basis for $\mathbf{V}$.

If $r < n$, we consider adding vectors from $\{e_1, \cdots, e_n\}$ to S to enlarge the set of independent vectors. We start with $\mathbf{e}_1$. If $\mathbf{e}_1$ is a linear combination of vectors in S, we remove e_1. Otherwise we add e_1 to S; in this case, the new set $S \cup \{\mathbf{e}_1\}$ is linearly independent and the number of vectors in this new set is $r + 1$. Since this new set is linearly independent, we have $r + 1 \leq n$. We move to $\mathbf{e}_2$ through $\mathbf{e}_n$ to arrive at a set with the number of vectors, which is $r + m$, $m \geq 1$. Let the final set be T.

This final set T is linearly independent. Moreover, every vector in $\{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ can be expressible as linear combinations of vectors in S,

$$\text{span}(T) = \mathbf{V}.$$

So T is a basis.

Theorem. Let W be a subspace of a finite dimensional vector space **V**, then

(a). W is finite dimensional.

(b). $\dim(W) \leq \dim(W)$.

(c). $W = V$ if and only if $\dim(W) = \dim(V)$.

Let $E = \{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ be a basis of the vector space.

Part (a). If W is zero subspace, then $\dim(W) = 0$. Otherwise, We run $\mathbf{e}_1$ through $\mathbf{e}_n$ to see whether some of them is in W. Suppose that $S = \{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_m}\} \in \mathbf{W}$ consider span(S). If span$(S) = W$, the dimension of W is m. If not, there exists $\mathbf{v} \in W \setminus \text{span}(S)$ and $v \neq 0$. Since $\mathbf{v}$ is not in span(S), we add $\mathbf{v}$ to S, and consider the spanning set of S and $\mathbf{v}$. The dimension of this new spinning set is $m + 1$, still denoted by S. We continue this process and arrive at

$$\text{span}(S) = W.$$

Part (b). By the proof of part (a), we have

$$\text{span}(S) = W.$$

So

$$\dim(W) \leq \dim(V).$$

Part (c). If $W = V$, then $\dim(W) = \dim(V)$. On the other hand, if $\dim(W) = \dim(V)$, denote this number by n. Suppose that $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ is a basis for W. Since $W \subset V$, this is also a linearly independent set in V. Since the number is n, by Theorem 4.5.4, we see that S is a basis for V. Hence $W = V$.

Homework and Reading.

Homework. Ex. # 2, # 4, # 6, # 7, # 8, # 9, # 12, # 14, # 15. True or false questions on page 217.

Reading. Section 4.6.