Lecture 21: Section 4.6

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Dimension

Let $\mathbf{V}$ be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ be a basis: B is linearly independent, and spans the vector space $\mathbf{V}$. The dimension of the vector space $\mathbf{V}$ is n.

Let B' be another basis for $\mathbf{V}$. The change of basis B to B' is related to a matrix of $n \times n$.

Theorem. From Section 4.4, let $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ be a basis for a finite dimensional vector space $\mathbf{V}$, and if

$$(\mathbf{v})_S = (c_1, c_2, \cdots, c_n)$$

is the coordinate vector of $\mathbf{v}$ relative to S, then the mapping

$$\mathbf{v} \to (\mathbf{v})_S$$

creates an one-to-one connection between vectors in the general vector space $\mathbf{V}$ and vectors in the vector space $\mathbb{R}^n$. We denote the coordinate vector by

$$(\mathbf{v})_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The change of basis problem.

Question. If $\mathbf{v}$ is a vector in a finite dimensional vector space $\mathbf{V}$, and if we change the basis for $\mathbf{V}$ from a basis B to a basis B', how is the coordinate vector $[\mathbf{v}]_B$ to $[\mathbf{V}]_{B'}$ related?

We take $\mathbb{R}^2$ as an example.

The change of basis in $\mathbb{R}^2$.

Let $\mathbf{B} = \{\mathbf{u_1}, \mathbf{u_2}\}$ be the old basis, and $\mathbf{B}' = \{\mathbf{u_1'}, \mathbf{u_2'}\}$ be new basis. Suppose that

$$u_1' = a\mathbf{u}_1 + b\mathbf{u}_2,$$
$$u_2' = c\mathbf{u}_1 + d\mathbf{u}_2.$$

Since we write $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u'}_1, \mathbf{u'}_2$ as column vectors, we have

$$\begin{bmatrix} u_1' & u_2' \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \times \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Let **v** any vector in $\mathbb{R}^2$. Write the coordinate vector **v** with respect to the basis B':

$$[\mathbf{v}]_{B'} = \begin{pmatrix} k_1' \\ k_2' \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = k_1' u_1' + k_2' u_2' = (k_1' a\mathbf{u}_1 + k_1' b\mathbf{u}_2) + (k_2' c\mathbf{u}_1 + k_2' d\mathbf{u}_2)$$

and

$$= (k_1' a + k_2' c)\mathbf{u}_1 + (k_1' b + k_2' d)\mathbf{u}_2.$$

Let $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be the coordinate vector of **v** with respect to B. Then

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_1' a + k_2' c \\ k_1' b + k_2' d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \times \begin{pmatrix} k_1' \\ k_2' \end{pmatrix}.$$

Solution to the change of basis.

Theorem. If we change the basis for a vector space $\mathbf{V}$ from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u'}_1, \mathbf{u'}_2, \cdots, \mathbf{u}'_n\}$, then for each vector $\mathbf{v}$ in $\mathbf{V}$, the old coordinate vector $[\mathbf{v}]_B$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ by the equation

$$[\mathbf{v}]_B = P_{B' \to B}[\mathbf{v}]_{B'},$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of P are

$$[u'_1]_B, [u'_2]_B, \cdots, [u'_n]_B.$$

Transition matrices.

The matrix P in Equation above is called the **transition matrix that maps B'-coordinate of v to B-coordinate of v**.

$$P_{B' \to B} = \left[[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \cdots, [\mathbf{u}'_n]_B \right],$$

where $[\mathbf{u}'_i]_B$ is the column vector expressing u'_i in terms of the basis $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$, and

$$P_{B \to B'} = [[\mathbf{u}_1]_{B'}, [\mathbf{u}_2]_{B'}, \cdots, [\mathbf{u}_n]_{B'}],$$

where $[\mathbf{u}_i]_B$ is the column vector expressing u_i in terms of the basis $\{\mathbf{u}'_1, \mathbf{u}'_2, \cdots, \mathbf{u}'_n\}$.

Example 1: finding the transition matrices.

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u'_1}, \mathbf{u'_2}\}$ for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1), \mathbf{u'}_1 = (1, 1), \mathbf{u'}_2 = (2, 1).$$

- **(a).** Find the transition matrix $P_{B' \to B}$ from B' to B.
- **(b).** Find the transition matrix $P_{B \to B'}$ from B to B'.

Solution.

Proof.

We write

$$\mathbf{u'}_1 = a\mathbf{u}_1 + b\mathbf{u}_2, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{u'}_2 = c\mathbf{u}_1 + d\mathbf{u}_2, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} = c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus $a = 1, b = 1$ and $c = 2, d = 1$. Thus the transition matrix $P_{B' \to B}$ is

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

□

Cont.

On the other hand, we write

$$\mathbf{u}_1 = \alpha \mathbf{u'}_1 + \beta \mathbf{u'}_2, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \gamma \mathbf{u'}_1 + \delta \mathbf{u'}_2, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \delta \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus $\alpha = -1, \beta = 1$, and $\gamma = 2, \delta = -1$. Thus the transition matrix $P_{B \to B'}$ is

$$\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Computing coordinate vectors.

Let B and B' be the bases in Example 1. Use an appropriate formula to find $\mathbf{v}_B$ given that $[\mathbf{v}]_{B'} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

Proof.

$$[\mathbf{v}]_B = P_{B' \to B}[\mathbf{v}]_{B'} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

□

Invertibility of transition matrices.

From Example 1,

$$P_{B' \to B} P_{B \to B'} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So the inverse of the matrix $P_{B \to B'}$ is $P_{B \to B'}$; conversely it is also true.

Theorem 4.6.1 If P is the transition matrix from a basis B' to a basis B for a finite dimensional vector space $\mathbf{V}$, then P is invertible, and the inverse P^{-1} is the transition matrix from B to B'.

Theorem. Let $B' = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}\}$ be any basis for the vector space $\mathbb{R}^n$ and let $S = \{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ be the standard basis for $\mathbb{R}^n$. If the vectors in these bases are written in column form, then

$$P_{B' \to S} = [\mathbf{u_1} | \mathbf{u_2} | \cdots | \mathbf{u_n}] .$$

The proof is clear from the definition.

Example.

Let $\mathbf{u}_1 = (1, 2, 1)$, $\mathbf{u}_2 = (2, 5, 0)$ and $\mathbf{u}_3 = (3, 3, 8)$, and $\mathbf{e}_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Since we can express the vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ in terms of $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$,

$$\begin{aligned}
\mathbf{u}_1 &= 1 \times \mathbf{e}_1 + 2 \times \mathbf{e}_2 + 1 \times \mathbf{e}_3, \\
\mathbf{u}_2 &= 1 \times \mathbf{e}_1 + 5 \times \mathbf{e}_2 + 0 \times \mathbf{e}_3, \\
\mathbf{u}_3 &= 3 \times \mathbf{e}_1 + 3 \times \mathbf{e}_2 + 8 \times \mathbf{e}_3,
\end{aligned}$$

we see that the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}.$$

Homework and Reading.

Homework. Ex. # 2, # 4, # 6, # 7, # 8, # 9, # 12, # 14, # 15. True or false questions on page 224.

Reading. Section 4.7.