Lecture 22: Section 4.7

Shuanglin Shao

December 2, 2013

Row Space, Column Space, and Null Space.

Definition. For an $m \times n$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

the vectors

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \mathbf{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ &\vdots \\ \mathbf{r}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

in $\mathbb{R}^n$ that are formed from the rows of A are called the **row vectors** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in $\mathbb{R}^m$ formed from the columns of A are called the column vectors of A.

Example.

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}.$$

The row vectors of A are

$$\mathbf{r}_1 = [2, 1, 0], \quad \mathbf{r}_2 = [3, -1, 4]$$

and the column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Def.

Definition. If A is an $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the **row space** of A, and the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the **column space** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of $\mathbb{R}^n$, is called the **null space** of A.

Theorem. A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Proof.

The linear system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

□

This linear system can be written as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

This proves the theorem.

Example 2.

Example 2. Let $Ax = \mathbf{b}$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}.$$

Solving the linear system by Gaussian elimination yields

$$x_1 = 2, x_2 = -1, x_3 = 3.$$

Thus from the theorem above,

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}.$$

Theorem. If x_0 is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

Conversely, for all choices of scalars $c_1, c_2, \cdots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

Bases for row spaces, column spaces, and null spaces.

Theorem. Elementary row operations do not change the null space of a matrix.

Theorem. Elementary row operations do not change the row space of a matrix.

Both proofs follow if we consider the three elementary row operations: exchange two rows, a nonzero constant multiplying a row, adding a multiple of a row to another row.

We just learned from the elementary row operations do not change the row space of a matrix. However the elementary row operations do change the column space of a matrix. For instance, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The column space is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Applying the third elementary row operations, we obtain

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The column space is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The spanning vectors are different, which generate different column spaces.

Finding a basis for the null space of a matrix.

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}.$$

Solving the equation $A\mathbf{x} = \mathbf{0}$, we obtain

$$\mathbf{x} = r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

These three vectors generate the null space.

Basis for row spaces and column spaces.

Theorem. If a matrix R is in **row echelon form**, then the row vectors with the leading $1'$ s (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading $1'$ s of the row vectors form a bassi for the column space of R.

Remark. Only when a matrix is in the row echelon form, the columns with leading $1'$ s of the row vectors form a basis for the column space. In general, it is not true for the original matrix since the elementary row operations will change the column space.

Example. Bases for row and column spaces.

The matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basis for the row space is

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \end{bmatrix},$$
$$\mathbf{r}_2 = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 \end{bmatrix},$$
$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The basis for the column space is

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Example: basis for a row space by row reduction.

Find a basis for the row space of a matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

By row reductions, we have

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basis for the row space is

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix},$$
$$\mathbf{r}_2 = \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix},$$
$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}.$$

We have talked about how to find a basis for the null space of a matrix by solving the linear system; how to find a basis for the row space by using the elementary row operations and select the rows with the leading 1's.

In the **remark above**: only when a matrix is in the row echelon form, the columns with leading $1'$ s of the row vectors form a basis for the column space. In general, it is not true for the original matrix since the elementary row operations will change the column space. We will see how to find a basis for the column space for a matrix.

Definition. If B is obtained from A by applying elementary row operations to A, then A and B are said to be **row equivalent.**

Theorem. If A and B are row equivalent matrices, then

(a). A given set of column vectors of A is linearly independent if and only if the corresponding column vectors are linearly independent.

(b). A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

Example. Find a basis by row reductions.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

By row reductions, we have

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The 1st, 3rd, 5th columns form a basis for the matrix R. By the theorem above, since A and R are row equivalent, the 1st, 3rd, 5th columns of the matrix A form a basis for A.

Basis for a vector space using row operations.

Find a basis for the subspace of $\mathbb{R}^5$ spanned by the vectors

$$\mathbf{v}_1 = (1, -2, 0, 0, 3), \mathbf{v}_2 = (2, -5, -3, -2, 6),$$

and

$$\mathbf{v}_3 = (0, 5, 15, 10, 0), \mathbf{v}_4 = (2, 6, 18, 8, 6).$$

We use the vectors form a matrix,

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}.$$

By using elementary row operations, it reduces to

$$R = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The **1st, 2nd, 3rd** rows form a basis for the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

Finding a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

consisting entirely of row vectors from A.

Basis for the row space of a matrix.

We know how to find a basis for the row space of a matrix by using the elementary row operations. The basis vectors come from the row echelon form of the original matrix. If we want to find the basis expressible by the original matrix. We use A^T.

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}.$$

By elementary row operations,

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the **1st, 2nd, 4th** columns of A^T form a basis for the column space of A^T. That is to say, the **1st, 2nd, 4th** rows of A form a basis for the row space of A.

Homework and Reading.

Homework. Ex. # 1, # 2 (b) (d), # 3(b), # 4, # 5 (b), # 6(b) (c), # 7 (b), # 9, #10 (c). True or false questions on page 237.

Reading. Section 4.8.