Lecture 23: Section 4.8

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December 4, 2013

Rank, Nullity, and the rank nullity theorem.

In the section, we will talk about the the rank of a matrix, the nullity of a matrix, and the rank-nullity theorem.

Theorem. The row space and column space of a matrix A have the same dimension.

Definition. This number is defined to be the rank of the matrix A. The dimension of the null space of a matrix A is called the nullity of the matrix.

Example: rank and nullity of a 4×6 matrix.

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}.$$

The reduced row echelon form is

$$A = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and second row form a basis for the row space; hence the rank of this matrix is 2.

The equation $A\mathbf{x} = \mathbf{0}$ implies that the following linear system,

$$\begin{cases} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 & = 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 & = 0. \end{cases}$$

Let $x_3 = s, x_4 = t, x_5 = p, x_6 = q$, then

$$x_1 = 4s + 28t + 37p - 13q,$$
$$x_2 = 2s + 12t + 16p - 5q.$$

Thus the solution space of $A\mathbf{x} = \mathbf{0}$, the null space, is spanned by 4 vectors.

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = s \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + p \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
$$

These four vectors are linearly independent. Thus the dimension of the null space, the nullity, is 4.

From the previous example, for the 4×6 matrix,

$$\text{rank}(A) + \text{nullity}(A) = 2 + 4 = 6.$$

Dimension Theorem.   If A is a matrix of $m \times n$,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

This is proven by reducing A to the reduced row echelon form. We omit the proof.

Example.

Find a basis for the row space of a matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}.$$

By row reductions, we have

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The dimension for the row space is 3, which is the rank of A. The nullity

$$\text{nullity}(A) = 6 - 3 = 3.$$

This is consistent with the number of free variables in the solution space of $A\mathbf{x} = \mathbf{0}$.

Maximum value for rank.

The rank of a matrix A_{mn} is the dimension of the row space,

$$\text{rank}(A) \leq m.$$

It is also the dimension of the column space,

$$\text{rank}(A) \leq n.$$

So it is less than both numbers, it is less than the smaller one:

$$\text{rank}(A) \leq \min(m, n).$$

Homework and Reading.

Homework. Ex. # 1, # 2 (b) (c) (d), # 5 (a) (b). True or false questions on page 247.

Reading. Section 5.1.