Lecture 24: Section 5.1

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In the section, we will talk about the the definitions of “eigenvalue” and “eigenvector” and discuss some of their basic properties.

Definition. If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an **eigenvector** if

$$A\mathbf{x} = \lambda \mathbf{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A and x is said to be an **eigenvector corresponding to** λ.

Let $A = I_n$. Then for any nonzero $\mathbf{x} \in \mathbb{R}^n$, then

$$I\mathbf{x} = 1 \cdot \mathbf{x}.$$

In this case, 1 is an eigenvalue, and $\mathbf{x}$ is an eigenvector.

Let

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda = 3.$$

Then

$$A\mathbf{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{x}.$$

In this case, 3 is an eigenvalue, and **x** is called an eigenvector.

Theroem 5.1.1 If A is an $m \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0.$$

This is called the **characteristic equation** of A.

Proof.

If $A\mathbf{x} = \lambda x$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$. Since this linear system has a nonzero solution, then the determinant of the coefficient matrix is zero. That is to say,

$$\det(\lambda I - A) = 0.$$

□

The previous equation tells us how to find eigenvalues. Once eigenvalues, we consider the null space determined by $(\lambda I - A)\mathbf{x} = \mathbf{0}$, where the nonzero vectors are eigenvectors.

Example. Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

Consider

$$\det(\lambda I - A) = 0.$$

Then

$$\begin{matrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{matrix} = (\lambda - 3)(\lambda + 1) = 0.$$

Thus

$$\lambda = 3, \quad \lambda = -1.$$

Example.

Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$.

Proof.

Consider

$$\det(\lambda I - A) = 0.$$

Then

$$\begin{matrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{matrix} \quad = \lambda^3 - 8\lambda^2 + 17\lambda - 4.$$

$\square$

The eigenvalues of A satisfies $\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$.
To solve this equation, we search for integer solution. This task can be simplified by exploiting the fact that all the integer solutions of a polynomial with the **integer coefficients**

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

must be divisors of the constant term c_n. Thus the only possible integer solutions of the equations are divisors of -4, that is, $\pm 1, \pm 2, \pm 4$.

By testing, 4 is an integer solution. We factor the equation

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0.$$

Thus $\lambda_1 = 4$, $\lambda_2 = 2 + \sqrt{3}$ and $\lambda_3 = 2 - \sqrt{3}$.

Eigenvalues for an upper triangular matrix and lower triangular matrix.

Find the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

where A is an upper triangular matrix and B is a lower triangular matrix.

For A,

$$\lambda\text{I-A} = \begin{matrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda - a_{nn} \end{matrix}$$

and it equals

$$(\lambda - a_{11})(\lambda - a_{12}) \times \cdots \times (\lambda - a_{nn}).$$

So the eigenvalues are the entires on the diagonal line,

$$a_{11}, a_{22}, \cdots, a_{nn}.$$

So is for the lower triangular matrix.

Theorem 5.1.3. If A is an $n \times n$ matrix, then the following statements are equivalent.

(a). λ is an eigenvalue of A.

(b). The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

(c). There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

(d). λ is a solution of the characteristic equation

$$\det(\lambda I - A) = 0.$$

Bases for eigenspaces.

Definition. The null space of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is called the eigen-space of A corresponding to λ.

We already computed the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$. For the eigenvalue $\lambda_1 = 3$,

$$(\lambda_1 I - A)\mathbf{x} = \begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So the solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

Thus this vector is a basis for the eigenspace of A corresponding to $\lambda = 3$.

For $\lambda = -1$,

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace of A corresponding to $\lambda = -1$.

Example: eigenvectors and bases for eigenspaces.

Find bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Proof.

The characteristic polynomial is

$$\det(\lambda I - A) = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0.$$

The factorization is $(\lambda - 1)(\lambda - 2)^2 = 0$. Thus the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$. So there are two eigenspacs of A.
By definition, for $\lambda = 2$,

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution can be expressed in terms of two parameters, By definition, for $\lambda = 1$,

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution can be expressed in terms of two parameters,

$$x_1 = -2s, x_2 = s, x_3 = s.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

This vector forms a basis for the eigen-space corresponding to $\lambda = 1$.

Powers of a matrix.

Theorem. If k is a positive integer, and λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

Proof.

This is proven by using the math induction. For $k = 1$, $A\mathbf{x} = \lambda x$. Suppose this is true for k, we need to prove it for $k + 1$. By hypothesis,

$$A^k\mathbf{x} = \lambda^k\mathbf{x}.$$

Then

$$A^{k+1}\mathbf{x} = A\left(\lambda^k\mathbf{x}\right) = \lambda^k A\mathbf{x} = \lambda^{k+1}\mathbf{x}.$$

This establishes the claim. □

In the previous example, we know that $\lambda = 1, 2$ are eigenvalues for A. What are the eigenvalues for A^7?

Proof.

The eigenvalues for A^7 is $2^7 = 128$ and $1^7 = 1$. The corresponding eigenvectors are the same. □

Homework and Reading.

Homework. Exercise. # 1, # 2, # 4 (a)(b), # 8 (a)(b). True or False questions on page # 304.