Lecture 4

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Matrix

Def. A **matrix** is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.

A general $m \times n$ matrix A can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where a_{ij} is called entries of matrix A. The size of the matrix is $m \times n$; m denotes the number of rows, n denotes the number of columns.

Examples of matrices.

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, 3 \times 2 \text{ matrix}.$$

Example.

$$A = \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, 1 \times 4 \text{ matrix}.$$

Example.

$$A = \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, 3 \times 3 \text{ matrix}.$$

Example.

$$A = \begin{bmatrix} 4 \end{bmatrix}, 1 \times 1 \text{ matrix}.$$

The matrix A can be written as

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$$

or

$$\begin{bmatrix} a_{ij} \end{bmatrix}$$

If we denote by a_j the j-th column of A, $1 \leq j \leq n$,

$$A = \begin{bmatrix} a_1 & a_2 & \cdots a_n \end{bmatrix},$$

where a_j is the column vector, $a_j := \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$. We can also write

$$A = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where b_i is the row vector, $1 \leq i \leq m$, $b_i := \begin{bmatrix} b_{i1} & b_{i2} \cdots & b_{in} \end{bmatrix}$.

Diagonal line of matrices

If the size of A is $n \times n$, A is said to be a square matrix. For a square matrix, the diagonal line is

$$\begin{bmatrix} a_{11} & * & \cdots & * \\ * & a_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & a_{nn} \end{bmatrix}.$$

If we add the entries on the diagonal line, the number

$$a_{11} + a_{22} + \cdots + a_{nn}$$

is called the trace of the matrix A, denoted by $\mathrm{tr}(A)$.

Equality of matrices

Def. Two matrices are said to be equal if they have the same size and their corresponding entries are equal.

Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}$. If $A = B$, then $x = 5$.

If $C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}$, $A \neq C$ because A and C have different sizes.

Addition and Subtraction of matrices.

Def. If A and B are matrices of the same size, then the sum $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the difference $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes can not be added or subtracted.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and

$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$, then

$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}$.

Scalar multiplication of matrices

Let c be a real number and

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

then

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}.$$

Examples

Example. Let $A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}$, then

$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix}$, and $-2A = \begin{bmatrix} -4 & -2 & 0 & -6 \\ 2 & 0 & -4 & -8 \\ -8 & 4 & -14 & 0 \end{bmatrix}$.

Multiplication of two matrices

Def. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products. In Notation, $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times r}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{r \times n}$, then

$$C = AB = \begin{bmatrix} c_{ij} \end{bmatrix}_{m \times n},$$

where $c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ir} b_{rj}$.

Examples.

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}_{3 \times 4}$. Since

the number of columns of A equals the number of the rows of B, the multiplication of AB is allowed. AB is a 2×4 matrix.

$$AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}.$$

We compute $c_{11} = \sum_{k=1}^{3} a_{1k} b_{k1} = 1 \times 4 + 2 \times 0 + 4 \times 2 = 12$, and $c_{12} = \sum_{k=1}^{3} a_{1k} b_{k2} = 1 \times 1 + 2 \times (-1) + 4 \times 7 = 27$. Similarly we compute the other entires:

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}.$$

Examples.

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}_{3\times 3} \text{ and } B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}_{3\times 3}$$. Since the

number of columns of A equals the number of the rows of B, the
multiplication of AB is allowed. AB is a 3×3 matrix.

$$AB = C = [c_{ij}].$$

We compute $c_{11} = 18 + 0 + 49 = 67$, $c_{12} = 41$, $c_{13} = 41$, $c_{21} = 64$,
$c_{22} = 21$, $c_{23} = 59$, $c_{31} = 63$, $c_{32} = 67$ and $c_{33} = 57$. To conclude

$$C = \begin{bmatrix} 67 & 41 & 41 \\ 64 & 21 & 59 \\ 63 & 67 & 57 \end{bmatrix}.$$

Partitioned matrices

Recall a matrix A of size $m \times n$ can be written as a column of row vectors:

$$A_{m \times n} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix},$$

where a_i are the row vectors of A. The matrix B of size $n \times p$ can be written as a row of column of vectors:

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix},$$

where b_j are the column vectors of B.

Partitioned matrices (Cont.)

We look at the product of AB. It is well-defined because A is of size $m \times n$ and B is of $n \times p$, and AB is of size $m \times p$. Similarly AB consists of the row vectors, too. For $1 \leq i \leq m$, The i-th row of AB is obtained by multiplying the i-th row of A with the matrix B, so the product of AB can be written

$$AB == \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix}.$$

Similarly the j-th column of AB is obtained by multiplying the matrix A with the j-th column of B; so

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}.$$

Examples.

The use of partitioned matrices in the product of AB has the advantage of singling out the rows or columns of AB. For instance, in the previous example, if $A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}_{3 \times 3}$ and $B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}_{3 \times 3}$. Then the second row of AB is $a_2 B$, i.e.,

$$\begin{bmatrix} 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 64 & 21 & 59 \end{bmatrix}.$$

Matrix products as linear combinations

Def. If $A_1, A_2, \cdots, A_r$ are matrices of the same size, and if $c_1, c_2, \cdots, c_r$ are scalars, then the expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_r A_r$$

is called a linear combination of $A_1, A_2, \cdots, A_r$ with coefficients $c_1, c_2, \cdots, c_r$.

Let A be an $m \times n$, and x an $n \times 1$ column:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}.$$

Then Ax is a matrix of $m \times 1$,

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Matrix form as a linear system

Consider a system of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{cases}.$$

This can be rewritten as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Coefficient matrix and augmented matrix

The system of linear equations can be written as

$$Ax = b,$$

where A is an $m \times n$ matrix and x, b are column vectors. A is called the **coefficient matrix** of the linear system. Recall the augmented matrix for the linear system is

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

This can be viewed as adjoining the column vector b to matrix A. We write

$$\left[\begin{array}{cc} A & |\, b \end{array} \right].$$

The advantage of writing it in this way will be clear in the next section, i.e., using the inverse of the matrix A.

Examples.

Example. For the linear system

$$\begin{cases} 3x_1 + 3x_2 + 3x_3 & = -3 \\ -x_1 - 5x_2 - 2x_3 & = 3 \\ -4x_2 + x_3 & = 0 \end{cases},$$

then $Ax = b$ can be written as

$$\begin{bmatrix} 3 & 3 & 3 \\ -1 & -5 & -2 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}.$$

Transpose of matrices

Def. If A is any $m \times n$ matrix, then the **transpose of A**, denoted by A^T, is defined to be the $n \times m$ that results by interchanging the rows and columns of A.

If A is an $m \times n$ matrix, A^T is a $n \times m$ matrix.

Examples.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4}$, then A^T is a 4×3 matrix.

The first row of A^T is the first column of A, etc. So

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}.$$

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.

Homework and Reading.

Homework. Ex.3 (b) (e)(h)(k). 4 (c)(f)(i)(l). 12 (a). 13(b). 16. 26. and the True-False exercise on page 38.

Reading. Section 1.4.