

Lecture 5

Shuanglin Shao

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1. Matrix notation and terminology

2. Inverse of matrices

Properties of matrix addition and scalar multiplication.

Theorem. Properties of matrix arithmetic. Assuming that the sizes of the matrices such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

- (a). $A+B=B+A$, commutative law for addition.
- (b). $A+(B+C) = (A+B) +C$, associative law for addition.
- (c). $A(BC) = (AB)C$, associative law for multiplication.
- (d). $A(B+C) = AB+AC$, left distribution law.
- (e). $(B+C) A = BA+CA$, right distribution law.
- (f). $A(B-C) = AB-AC$.

Cont.

$$(g). (B-C) A = BA - CA.$$

$$(h). a(B+C) = aB + aC.$$

$$(i). a(B-C) = aB - aC.$$

$$(j). (a+b) C = aC + bC.$$

$$(k). (a-b) C = aC - bC.$$

$$(l). a(bC) = (ab) C.$$

$$(m). a(BC) = (aB) C = B(aC).$$

We skip the proofs of these identities.

Examples.

Example. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix},$$

and

$$BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}.$$

Cont.

Thus

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix},$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}.$$

Therefore $(AB)C = A(BC)$. This is an example of the associative law for matrix multiplication.

commutative law in matrix multiplication not always true

Let $a, b \in \mathbb{R}$. It is always true that $ab = ba$. That is to say, the commutative law always holds for real numbers. However it is not the case for matrices.

Example. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then for any $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ and } BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Thus $AB = BA$.

Example. Let $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}, \text{ and } BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}.$$

This shows that

$$AB \neq BA.$$

Another examples that commutative laws for matrices fails could be found by considering the sizes. For instance, if A is 2×3 matrix, B is 3×3 matrix, then AB is a 2×3 matrix. However, BA is not well-defined.

Zero matrices

Def. The zero matrix is the matrix whose entries are all zero. We denote by 0 the zero matrix.

Example.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Properties of zero matrices.

Theorem. Let A be of size such that all operations of matrices can be performed.

(a). $A + 0 = 0 + A = A$.

(b). $A - 0 = A$.

(c). $A - A = A + (-A) = 0$.

(d). $0 A = 0$.

(e). If $cA = 0$, then either $c = 0$ or $A = 0$.

We give a proof of **(e)**. For simplicity, let us assume that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Then

$$cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}.$$

Since $cA = 0$, then all the entries of $cA = 0$:

$$ca_{11} = 0, ca_{12} = 0, ca_{21} = 0, ca_{22} = 0.$$

Inspecting the four identities, if $c = 0$, all four holds; if $c \neq 0$, then

$$a_{11} = a_{12} = a_{21} = a_{22} = 0.$$

This implies that $A = 0$.

Failure of the cancellation law for matrices.

The claim of **(e)** in previous theorem says that the cancellation law for scalar multiplication of matrices hold. This is also true for multiplication of two real numbers: if $a, b \in \mathbb{R}$,

$$ab = 0, \Rightarrow a = 0, \text{ or } b = 0.$$

However it is not always the case for multiplication of two matrices.

Cont.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -4 \\ 0 & 0 \end{bmatrix}$. Then

$$AB = 0.$$

But neither $A = 0$ nor $B = 0$.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$. Then

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}.$$

Even though $AB = AC$, we can not cancel out A to obtain $B = C$.
Actually $B \neq C$.

Identity matrix

Def. A square matrix with 1's on the main diagonal line and zeros elsewhere is called an identity matrix. Denote by I_m the identity matrix of size $m \times m$.

Examples.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Properties of identity matrices

Let A be of size $m \times n$. Then

$$I_m A = A, A I_n = A.$$

Examples.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},$$

and

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Inverse matrix

Def. Let A, B be of square matrices of size $n \times n$.

- ▶ If $AB = BA = I_n$, then B is the **inverse matrix** of A , and A is also the **inverse matrix** of B . In this case, we say that A is invertible; B is invertible.
- ▶ If no such matrix B exists, A is said to be **singular**.

Examples.

Example. Let $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

and

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By definition, A and B are inverse of each other.

Examples.

Example. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$. For $B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 2a_{11} & 2a_{12} \end{bmatrix}.$$

If AB is an identity matrix, $a_{11} = 1, a_{12} = 1$ from the main diagonal line. But AB then reduces to $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, which is not an identity matrix. So A is a singular matrix.

Inverse of 2×2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$. Then the inverse of A , which we denote by A^{-1} , is

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. Compute

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Examples.

Example. Let $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$. Then

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ -5/7 & 6/7 \end{bmatrix}.$$

Solution of a linear system by matrix inversion

Consider a linear system

$$\begin{cases} ax + by = u, \\ cx + dy = v \end{cases}$$

with the condition that $ad - bc \neq 0$, find the solutions.

Proof. Write the system into the matrix form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}, \text{ i.e., } A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

Then from the condition $ad - bc \neq 0$, A is invertible. We multiply A^{-1} to both sides, then

$$A^{-1}A \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} u \\ v \end{bmatrix}, \text{ i.e., } I_2 \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Thus the solution is given by $A^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$. Since

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{du-bv}{ad-bc} \\ \frac{av-cu}{ad-bc} \end{bmatrix}.$$

The inverse of products

If A and B are invertible matrices of size $n \times n$, then the product AB is invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. We verify

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n;$$

also

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

So AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Examples.

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}.$$

By the formula of computing the inverse of 2×2 matrix,

$$(AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}.$$

On the other hand, $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$.

Then

$$B^{-1}A^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}.$$

Thus this examples shows that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Theorem on inverses of matrices.

Def. Let n be a nonnegative integer. Define $A^n = A \times A \times \cdots \times A$, n times. When $n = 0$, $A^0 = I_n$.

Theorem. If A is invertible, n is a nonnegative integer, then

- ▶ A^{-1} is invertible, and $(A^{-1})^{-1} = A$.
- ▶ A^n is invertible, and $(A^n)^{-1} = (A^{-1})^n = A^{-1} \times A^{-1} \times \cdots \times A^{-1}$, n times. That is to say, the operations of taking powers and inverses of a matrix is commutative.
- ▶ kA is invertible for any nonzero scalar k , and $(kA)^{-1} = k^{-1}A^{-1}$.
- ▶ $(A^T)^{-1} = (A^{-1})^T$: the operation of taking inverse and transpose of a matrix is commutative.

The proof is omitted.

Example.

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. Then

$$(A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}.$$

On the other hand,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix};$$

then

$$(A^3)^{-1} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Thus

$$(A^3)^{-1} = (A^{-1})^3.$$

Matrix polynomials

If A is a square matrix of $n \times n$, and if

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m,$$

then

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_mA^m.$$

Examples.

Let $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$ and $p(x) = x^2 - x - 3$. Then

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Properties of transposes.

If the sizes of the matrices are such that the stated operations can be performed, then

- ▶ $(A^T)^T = A$.
- ▶ $(A + B)^T = A^T + B^T$.
- ▶ $(A - B)^T = A^T - B^T$.
- ▶ $(kA)^T = kA^T$.
- ▶ $(AB)^T = B^T A^T$.

Inverse of transposes.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$. We know that

$$(A^T)^{-1} = (A^{-1})^T.$$

By computing $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, we see that

$$(A^T)^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Homework and Reading.

Homework. Ex. #4, #6, #8, #14, #18 (a) (d) (e). #27, #30, #36, #40, #53, #54, and the True-False exercise on page 50 .

Reading. Section 1.5.