

# Lecture 6 & 7

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1. Elementary matrices

2. Equivalence Theorem.

3. A method of inverting matrices

**Def.** An  $n \times n$  matrix is called an elementary matrix if it can be obtained from the  $n \times n$  identity matrix  $I_n$  by performing a single elementary row operation.

Recall

- ▶ Multiply a row by a nonzero constant  $c$ .
- ▶ Interchange two rows.
- ▶ Add a constant  $c$  times one row to another.

## Examples.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \text{ times 2nd row by } -3.$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{ interchange the 2nd and 4th rows .}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ add 3 times 3rd row to 1st row.}$$

## Inverse row operations.

Let  $E$  be an elementary matrix. By definition,  $E$  is obtained by performing a single row operation on the identity matrix  $I_n$ . We can invert this row operation and recover the identity matrix  $I_n$  from  $E$ . **This inversion is also an elementary row operation.**

**(1).** If  $E$  is obtained from  $I_n$  by multiplying a row of  $I_n$  by  $c \neq 0$ , then multiplying the same row by  $\frac{1}{c}$ , we obtain  $I_n$ .

**Example.**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\times 2 \text{ to the 1st row}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\times \frac{1}{2} \text{ the 1st row}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(2). If  $E$  is by obtained from  $I_n$  by exchanging two rows of  $I_n$ , then exchanging the same two rows of  $E$ , we obtain  $I_n$ .

**Example.**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{exchange 1st and 2nd}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{exchange 1st and 2nd}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(3). If  $E$  is by obtained from  $I_n$  by adding  $c$  times a row to another row, then adding  $-c$  times the same row to the same row, we obtain  $I_n$ .

**Example.**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{adding 2 times 2nd row to the first}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ \xrightarrow{\text{adding -2 times 2nd row to the first}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

# Elementary row operations $\Leftrightarrow$ Multiplying elementary row matrices

**Theorem.** If the elementary matrix  $E$  results from performing an elementary row operation on  $I_m$  and if  $A$  is an  $m \times n$  matrix, then the **left multiplication** product  $EA$  is the matrix obtained from performing the same row operation on  $A$ .

## Proof.

We split the proof into three cases according to the three elementary row operations.

**Step 1.** If  $E$  is obtained by multiplying the  $i$ -th row of the identity matrix  $I_m$  by a nonzero constant  $c$ . Then we write  $E$  in the row vectors.

$$E = \begin{bmatrix} e_1 \\ \vdots \\ ce_j \\ \vdots \\ e_m \end{bmatrix}, \Rightarrow EA = \begin{bmatrix} e_1 A \\ \vdots \\ ce_j A \\ \vdots \\ e_m A \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ ca_j \\ \vdots \\ a_m \end{bmatrix},$$

where  $e_i$  denotes the a row of  $m$  entries with the only  $i$ -th position is 1, the other entries are 0;  $a_j$  denotes the  $i$ -th row of  $A$ . Thus  $EA$  is the same as multiplying the  $i$ -th row of  $A$  by  $c$ . Note that this procedure is invertible: if we multiply the  $i$ -th row of  $A$  by  $c$ , it is the same as left multiplying  $A$  by  $E$ .

**Step 2.** If  $E$  is obtained by exchanging the  $i$ -th and the  $j$ -th row of the identity matrix.

$$E = \begin{bmatrix} \vdots \\ e_j \\ \vdots \\ e_i \\ \vdots \end{bmatrix}, \Rightarrow EA = \begin{bmatrix} \vdots \\ e_j A \\ \vdots \\ e_i A \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \end{bmatrix}.$$

Thus  $EA$  is the same as exchanging the  $i$ -th and the  $j$ -th row of  $A$ .  
The converse is also true.

**Step 3.** If  $E$  is obtained by adding  $c$  times  $i$ -th to the  $j$ -th row of the identity matrix.

$$E = \begin{bmatrix} \vdots \\ e_i \\ \vdots \\ ce_i + e_j \\ \vdots \end{bmatrix}, \Rightarrow EA = \begin{bmatrix} \vdots \\ e_i A \\ \vdots \\ ce_i A + e_j A \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ a_i \\ \vdots \\ ca_i + a_j \\ \vdots \end{bmatrix}.$$

Thus  $EA$  is the same as adding  $c$  times  $i$ -th to the  $j$ -th row of  $A$ . The converse is also true. This finishes the proof.

## Examples.

Consider

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix},$$

and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Then

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}.$$

**Theorem.** Every elementary matrix is invertible, and the inverse is also an elementary matrix.

**Proof.** Let  $E$  be an elementary matrix:  $E$  is obtained from  $I_n$  by performing an elementary row operation. By the discussion above, we can recover  $I_n$  from  $E$  by another elementary row operation. By the Theorem above, this another elementary operation corresponds to an elementary matrix  $E_0$  such that

$$E_0E = I_n.$$

Similarly

$$EE_0 = I_n.$$

Hence  $E$  and  $E_0$  are invertible to each other.

**Theorem.** If  $A$  is an  $n \times n$  matrix, then the following statements are equivalent, that is, all true or false.

- (a).  $A$  is invertible.
- (b).  $Ax = 0$  has only the trivial solution.
- (c). The reduced row echelon form of  $A$  is  $I_n$ .
- (d).  $A$  is expressible as a product of elementary matrices.

**Proof.** We prove the chain of implications,  $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$ .

**Step 1. (a)  $\Rightarrow$  (b).** For  $Ax = 0$ , multiplying both sides by  $A^{-1}$ ,

$$A^{-1}Ax = A^{-1}0, \Rightarrow x = 0.$$

So the system  $Ax = 0$  has only the trivial solution.

**Step 2. (b)  $\Rightarrow$  (c).** We prove (c) by contradiction. If the reduced row echelon form  $B$  of  $A$  is not  $I_n$ ,  $B$  contains at least a row of consisting of entirely zero. Indeed, if each row contains at least one nonzero number, this number is 1. Since  $B$  is in the reduced row echelon form, no two 1's are not in the same column. Since the matrix  $A$  is of  $n \times n$ ,  $B$  is  $I_n$ . The augmented matrix is  $Ax = 0$  is

$$[A | 0].$$

## Cont.

After performing a sequence of elementary row operations, we obtain

$$[B, | 0].$$

Then one row above consists of entirely zero, which implies that there are infinitely many solutions to the system. A contradiction. This proves **(c)**.

**Step 3. (c)  $\Rightarrow$  (d).** Since  $I_n$  is the reduced row echelon form of  $A$ , and the reduced row echelon form is obtained from  $A$  by a sequence of elementary row operations, so there exists a corresponding sequence of elementary matrices  $E_1, E_2, \dots, E_m$  such that

$$I_n = E_m E_{m-1} \times \cdots \times E_1 A.$$

Since each  $E_i$  are invertible, whose corresponding inverse is denoted by  $F_i$ ,  $1 \leq i \leq m$ ,

$$F_1 \times \cdots \times F_m I_n = (F_1 \times \cdots \times F_m)(E_m E_{m-1} \times \cdots \times E_1 A) = A.$$

So

$$A = F_1 \times \cdots \times F_m.$$

This proves **(d)**.

**Step 4. (d)  $\Rightarrow$  (a).** Suppose  $A = F_1 \times \cdots \times F_m$ , where  $\{F_i\}$  is elementary matrix. The inverse of each elementary matrix  $F_i$  is also an elementary matrix  $E_i$ . So let  $B = E_m \times \cdots \times E_1$ . Then

$$AB = I_n$$

and

$$BA = I_n.$$

So  $A$  is invertible, and the inverse of  $A$  is  $B$ .

## Computing $A^{-1}$ by elementary matrices.

**Inversion Algorithm.** To find the inverse of an invertible matrix  $A$ , find a sequence of elementary row operations that reduces  $A$  to the identity and then perform that same sequence of operations on  $I_n$  to obtain  $A^{-1}$ .

**Proof.** Suppose that the sequence of elementary row operations that reduce  $A$  to  $I_n$  is  $E_1, E_2, \dots, E_k$ , i.e.,

$$E_k \times E_{k-1} \times \cdots \times E_1 A = I_n.$$

Thus multiplying both sides by  $A^{-1}$  from the right, we see that

$$E_k \times E_{k-1} \times \cdots \times E_1 = A^{-1},$$

i.e.,

$$E_k \times E_{k-1} \times \cdots \times E_1 I_n = A^{-1}.$$

This proves the theorem.

## Example.

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}.$$

The way to adjoin  $I_3$  to  $A$ :

$$[A|I_3].$$

Then we find  $A^{-1}$  is to perform the elementary row operations that reduce  $A$  to  $I_3$  on the identity matrix  $I_3$ .

$$[A|I_3] \Rightarrow [I_3|A^{-1}].$$

$$[A|b] = \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$
$$\Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right].$$

Thus

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

## Example.

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}.$$

Apply the previous procedure,

$$[A|I_3] = \left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array} \right].$$

After the elementary row operations,  $A$  can not be reduced to the identity matrix  $I_3$ . Hence the matrix  $A$  is not invertible.

# Homework and Reading.

**Homework.** Ex. #3, #4, #10, #14, #28, #30, #37, and the True-False exercise on page 60 .

**Reading.** Section 1.6.