Lecture 8

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Number of solutions of a linear system.

Theorem. A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Proof. Let the linear system be $Ax = b$. Naturally there are 3 cases: (a) no solutions. (b) only one solution. (c). More than one solution, i.e., at least two solutions. We prove in the case (c), there are actually infinitely many solutions. Let u, v be two different solutions to $Ax = b$. Then

$$Au = b,$$
$$Av = b.$$

Then by linearity of A,

$$A(u - v) = 0.$$

Since $u \neq v$, $u - v$ not identically zero solution.

Cont.

So for $k \in \mathbb{N}$, $u + k(u - v)$ is a solution to $Ax = b$:

$$A(u + k(u - v)) = Au + kA(u - v) = x + 0 = x.$$

For the infinitely choices of k in $\mathbb{N}$, there are infinitely many different $u + k(u - v)$. So we prove (c).

Solving linear system by matrix inversion.

Theorem. If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix b, then system of equations $Ax = b$ has exactly one solution, namely $x = A^{-1}b$.

Proof. Firstly $A^{-1}b$ is a solution because $A(A^{-1}b) = AA^{-1}b = I_n b = b$. So we need to prove that it is the only solution. If v is any solution to $Ax = b$, then

$$Av = b.$$

Multiplying both sides by A^{-1},

$$v = A^{-1}b.$$

Thus we prove the theorem.

Examples.

Consider

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = 5, \\ 2x_1 + 5x_2 + 3x_3 & = 3, \\ x_1 + 8x_3 & = 17. \end{cases}$$

This linear system can be written as $Ax = b$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}.$$

The inverse matrix of A:

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

So by the theorem above,

$$x = A^{-1}b = A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Linear systems with a common coefficient matrix.

Suppose we solve a sequence of linear systems with a common coefficient matrix:

$$Ax = b_1,\ Ax_2 = b_2, \cdots, Ax = b_k.$$

(1). If A is invertible, then the solutions to each system is easier.

$$x_1 = A^{-1}b_1,\ x_2 = A^{-1}b_2, \cdots, x_k = A^{-1}b_k.$$

(2). If A is not known to be invertible, then adjoin the columns

to the matrix A,

$$\left[\ A|\ b_1|\ b_2|\ \cdots|\ b_k\ \right].$$

We apply the Gaussian-Jordan elimination to reduce the matrix A to the reduced row echelon form. In this way, we can solve the linear system at once.

Solving two linear systems at once: Example.

Solve the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = 4, \\ 2x_1 + 5x_2 + 3x_3 & = 5, \\ x_1 + 8x_3 & = 9. \end{cases} \quad \begin{cases} x_1 + 2x_2 + 3x_3 & = 1, \\ 2x_1 + 5x_2 + 3x_3 & = 6, \\ x_1 + 8x_3 & = -6. \end{cases}.$$

We will follow the method outlined above to solve these two systems.

Cont.

$$\begin{bmatrix} -40 & 16 & 9 & |4 & |1 \\ 13 & -5 & -3 & |5 & |6 \\ 5 & -2 & -1 & |9 & |-6 \end{bmatrix}.$$

Reduce this matrix to the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & |1 & |2 \\ 0 & 1 & 0 & |0 & |1 \\ 0 & 0 & 1 & |1 & |-1 \end{bmatrix}.$$

So $x_1 = 1, x_2 = 0, x_3 = 1$ is a solution to the first system;
$x_1 = 2, x_2 = 1, x_3 = -1$ is a solution to the second system.

Properties of invertible matrices.

Recall the definition of invertible matrices: A is an invertible matrix if and only if there exists a matrix B such that

$$AB = I_n, \text{ and } BA = I_n.$$

That is to say, we need to verify two conditions to say that A and B are invertible. In fact, for square matrices, just one condition suffices.

Theorem. Let A be a square matrix. Then

(a). If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$;

(b). If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

Proof. We prove that **(a)** holds. The proof for **(b)** is similar.

Step 1. We first show that A is invertible by showing that the linear system $Ax = 0$ has only trivial solution, $x = 0$. To $Ax = 0$, multiplying both sides by B,

$$B(AI) = B0 = 0, \Rightarrow (BA)x = 0, \Rightarrow x = 0.$$

Thus by the equivalence theorem in Section 1.5, we see that A is invertible.

Step 2. Since A is invertible, multiplying $BA = I$ by A^{-1}, we obtain

$$(BA)A^{-1} = IA^{-1} = A^{-1}.$$

Thus

$$B = A^{-1}.$$

We apply a sequence of elementary row operations to B to obtain the reduced row echelon form.

$$E_1 \times \cdots \times E_k B = \begin{bmatrix} I_r \, P \\ 0 \, 0 \end{bmatrix},$$

where $r \leq n$ and P is a matrix of $r \times (n-r)$. Then apply $E_1, \cdots, E_k$ to $BA = I$ and obtain

$$\begin{bmatrix} I_r \, P \\ 0 \, 0 \end{bmatrix} A = E_1 \cdots E_k.$$

Since the left hand side is expressible in terms of a product of elementary matrices, it is invertible. However, the left hand side is a matrix whose last $(n-r)$ rows are identically zero. It is not invertible. A contradiction.

Equivalent statements.

If A is an $n \times n$ matrix, then the following are eauivalent.

(a). A is invertible.

(b). $Ax = 0$ has only the trivial solution.

(c). The reduced row echelon form of A is I_n.

(d). A is expressible as a product of elementary matrices.

(e). $Ax = b$ is consistent for every $n \times 1$ matrix.

(f). $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

We already proved that $(a) - - - (d)$ are equivalent. So it suffices to prove that $(a) \Rightarrow (f) \Rightarrow (e) \Rightarrow (a)$.

Step 1. $(a) \Rightarrow (f) \Rightarrow (e)$. This is easy. We omit the proof.

Step 2. $(e) \Rightarrow (a)$. For every column vector b, the linear system $Ax = b$ always has a solution. Let

$$b_i = \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix},$$

where except for the i-th position is 1, the other entries are 0. Let x_i be the solution to the linear system $Ax = b_i$. We construct a matrix

$$\begin{bmatrix} Ax_1 & Ax_2 \cdots Ax_n \end{bmatrix}.$$

This is precisely

$$AC = \begin{bmatrix} Ax_1 & Ax_2 \cdots Ax_n \end{bmatrix} = I_n,$$

where C is a matrix whose columns are x_i. Thus $AC = I_n$. So A is invertible.

Theorem. Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Proof. This is clear from the equivalence of (e) and (a).

Example: determining consistency by elimination.

What conditions must b_1, b_2 and b_3 satisfy in order for the system of equations

$$\begin{cases} x_1 + x_2 + 2x_3 & = b_1 \\ x_1 + x_3 & = b_2 \\ 2x_1 + x_2 + 3x_3 & = b_3 \end{cases}$$

to be consistent.

The augmented matrix is

$$\left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array}\right].$$

We apply the elementary row operations:

$$\left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right].$$

So the system is consistent if and only if $b_3 - b_1 - b_2 = 0$, i.e., $b_3 = b_1 + b_2$.

Example: determining consistency by elimination.

What conditions must b_1, b_2 and b_3 satisfy in order for the system of equations

$$\begin{cases} x_1 + 2x_2 + 2x_3 & = b_1 \\ 2x_1 + 5x_2 + 3x_3 & = b_2 \\ x_1 + 8x_3 & = b_3. \end{cases}$$

to be consistent. The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{bmatrix}.$$

We apply the elementary row operations to obtain

$$\begin{bmatrix} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{bmatrix}.$$

So the solution is given by

$$
\begin{aligned}
x_1 &= -40b_1 + 16b_2 + 9b_3 \\
x_2 &= 13b_1 - 5b_2 - 3b_3 \\
x_3 &= 5b_1 - 2b_2 - b_3.
\end{aligned}
$$

Homework and Reading.

Homework. Ex. #2, #6,#9, #10, #14, #16, #18 , and the True-False exercise on page 66.

Reading. Section 2.1.