

# Lecture 9

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# Determinants

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then  $A$  is invertible if and only if  $ad - bc \neq 0$ . The number  $ad - bc$  is called the determinant of the matrix  $A$ , denoted by

$$\det(A) = ad - bc, \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

# Minors and cofactors

**Def.** If  $A$  is a square matrix, then the minor of entry  $a_{ij}$  is denoted by  $M_{ij}$  and is defined to be the determinant of the submatrix that remains after the  $i$ -th row and the  $j$ -th column are deleted from  $A$ . The number  $(-1)^{i+j}M_{ij}$  is denoted by  $C_{ij}$  and is called the cofactor of entry  $a_{ij}$ .

## Finding minors and cofactors

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}.$$

The minor of entry  $a_{11}$  is

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 5 \times 8 - 6 \times 4 = 16.$$

The cofactor of  $a_{11}$  is

$$C_{11} = (-1)^{1+1} M_{11} = 16.$$

Similarly the minor of entry  $a_{32}$  is

$$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26.$$

The cofactor of  $a_{32}$  is

$$C_{32} = (-1)^{3+2} M_{32} = -26.$$

## Cofactor expansion of a $2 \times 2$ matrix

For  $2 \times 2$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

we have

$$C_{11} = M_{11} = a_{22}, C_{12} = -M_{12} = -a_{21}, C_{21} = -a_{12}, C_{22} = M_{22} = a_{11}.$$

Thus we verify the following identities hold:

$$\begin{aligned} \det(A) &= a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22}. \end{aligned}$$

This is an example of expanding the determinants along rows or columns.

## Definition of general determinants.

**Definition.** If  $A$  is an  $n \times n$  matrix, then the number obtained by multiplying the entries in any row or column of  $A$  by the corresponding cofactors and adding the resulting products is called the determinant of  $A$ , and the sums themselves are called cofactor expansion of  $A$ , that is

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj},$$

**i.e., cofactor expansion along the  $j$ -th column.**

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

**i.e., cofactor expansion along the  $i$ -th row.**

# The cofactor expansion independent of row/column choices

For a square matrix, the number  $\det(A)$  is independent of the choices of rows or columns. It is due to the following

**Theorem.** If  $A$  is an  $n \times n$  matrix, then regardless of which row or column of  $A$  is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting product is always the same.

The proof of this theorem is skipped.

An example is the cofactor expansion of a  $2 \times 2$  matrix above.

## Examples.

**Example.** Cofactor expansion along the first row. Let

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} \\ &\quad - \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} \\ &= -12 + 11 + 0 = -1. \end{aligned}$$

## Examples.

**Example.** Cofactor expansion along the first column. Let

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} \\ &\quad - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} \\ &= -12 - 4 + 15 = -1. \end{aligned}$$

## smart choice of row or column

Let

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

To find  $\det(A)$ , it is easy to use the cofactor expansion along the row or columns that contains most zeros. So we will expand along the second column.

$$\det(A) = 1 \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For this determinant of  $3 \times 3$  matrix, it is easy to expand it along the second column.

$$\begin{aligned} \det(A) &= (-2) \times \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ &= -6. \end{aligned}$$

# Determinant of a lower triangular matrix

Let

$$\det(A) = \begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

We continuously expand it along the first rows.

$$\begin{aligned} \det(A) &= a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \\ &= a_{11} a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix} \\ &= a_{11} a_{22} a_{33} a_{44}. \end{aligned}$$

## Theorem.

It is easy to generalize the example above to the following theorem.

**Theorem.** If  $A$  is an  $n \times n$  triangular matrix (upper triangular, lower triangular, or diagonal), then  $\det(A)$  is the product of the entries on the main diagonal of the matrix, that is

$$\det(A) = a_{11}a_{22} \cdots a_{nn}.$$

# Homework and Reading.

**Homework.** Ex. #4, #6, #16, #22, #33, #36, and the True-False exercise on page 100.

**Reading.** Section 2.2.