Math 290, Midterm II-key

Name (Print): (first)____________ (last)____________

Signature:

The following rules apply:

- There are a total of 20 points on this 50 minutes' exam. This contains 7 pages (including this cover page) and 9 problems. Check to see if any page is missing. Enter all requested information on the top of this page.
- To get full credit for a problem, you must show the details of your work in a reasonably neat and coherent way and in the space provided. Answers unsupported by an argument will get little credit.
- NO books. No computers. No cell phones. Do all of your calculations on this test paper.

Multiple choice problem	1	2	3	4	5	6	7	8
Answer	C	B	A	A	A	B	B	C

Problem 1. (2 points). Determine which of the following statements is true.

(A). Two equivalent vectors must have the same initial point.

(B). The vectors (a, b) and $(a, b, 0)$ are equivalent.

(C). If a and b are nonzero scalars such that $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors.

(D). If k and m are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors, then

$$(k + m)(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + m\mathbf{v}.$$

(E). The linear combinations $a_1\mathbf{v_1} + a_2\mathbf{v_2} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ are equal if and only if $a_1 = b_1, a_2 = b_2$.

Problem 2. (2 points). Determine which of the following statements is true.

(A). Let $\mathbf{u} = (-3, 2, 1, 0)$, $\mathbf{v} = (4, 7, -3, 2)$ and $\mathbf{w} = (5, -2, 8, 1)$ and $\mathbf{2u} + \mathbf{x} = \mathbf{v} + \mathbf{w}$. Then

$$x = (15, 2, 3, 3).$$

(B). The scalar (c_1, c_2, c_3) such that

$$c_1(1, 2, 0) + c_2(2, 1, 1) + c_3(1, 3, 2) = (1, 1, 1)$$

is $c_1 = -\frac{1}{7}, c_2 = \frac{3}{7}, c_3 = \frac{2}{7}$.

(C). If each component of a vector in $\mathbb{R}^3$ is doubled, the norm of the vector is the same.

(D). If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(E). If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

Problem 3. (2 points.) Determine which of the following statements is true.

(A). The Cauchy-Schwarz inequality says that, if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|,$$

where $\mathbf{u}\cdot\mathbf{v}$ denotes the inner product in $\mathbb{R}^n$.

(B). The triangle inequality states that, if $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^3$, then

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

If $\mathbf{u}$ and $\mathbf{v}$ are collinear, then $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ holds.

(C). If $\mathbf{u}$ and $\mathbf{a}$ are vectors in $\mathbb{R}^3$ and $\mathbf{a} \ne \mathbf{0}$, then the projection of $\mathbf{u}$ along $\mathbf{a}$ is $\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|}\mathbf{a}$.

(D). Let $\mathbf{u}, \mathbf{v}$ are two nonzero vectors in $\mathbb{R}^2$. Then $\mathbf{u}+\mathbf{v}$ has the same direction either as that of $\mathbf{u}$ or $\mathbf{v}$.

(E). For all vectors $\mathbf{u}$ and $\mathbf{v}$, it is true that $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

Problem 4. (2 points.) Determine which of the following statements is wrong.

(A). If the relationship

$$\text{proj}_{\mathbf{a}}\,\mathbf{u} = \text{proj}_{\mathbf{a}}\,\mathbf{v}$$

holds for some nonzero vector $\mathbf{a}$, then

$$\mathbf{u} = \mathbf{v}.$$

(B). Let L be the line in $\mathbb{R}^2$ that contains the point $(1,2)$, and is parallel to the nonzero vector $(2,1)$. Then the equation of the line in the parameter form is

$$(1+2t, 2+t)$$

where t is a parameter in $\mathbb{R}$.

(C). If A is an $m\times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.

(D). Let A be an $m\times n$ matrix and let $\mathbf{V}$ be the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consisting of all vectors in $\mathbb{R}^n$ with the standard addition and scalar multiplication in $\mathbb{R}^n$. Then $\mathbf{V}$ is a vector space.

(E). If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the non-homogeneous linear system $A\mathbf{x} = b$, then $\mathbf{x_1} - \mathbf{x_2}$ is a solution of the corresponding homogeneous linear system.

Problem 5. (2 points.) Determine which of the following statements is true.

(A). The set of all pairs of real numbers of the form $\{(x, 0) : x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard operations in $\mathbb{R}^2$ is a vector space.

(B). The set of all pairs of real numbers of the form $\{(x, y) : x \geq 0, y \in \mathbb{R}\}$ with the standard operations in $\mathbb{R}^2$ is a vector space.

(C). The set of invertible matrices with the standard matrix addition and scalar multiplication is a vector space.

(D). The set of invertible matrices in the matrix space M_{22} is a subspace.

(E). The set $\mathbb{R}^2$ is a subspace in $\mathbb{R}^3$.

Problem 6. (2 points.) Determine which of the following statements is wrong.

(A). The intersection of any two subspaces of a vector space $\mathbf{V}$ is a subspace of $\mathbf{V}$.

(B). The union of any two subspaces of a vector space $\mathbf{V}$ is a subspace.

(C). The vectors $\mathbf{v}_1 = (1, 1, 0), \mathbf{v}_2 = (1, 0, 1)$ and $\mathbf{v}_3 = (0, 1, 1)$ span $\mathbb{R}^3$.

(D). All the linear combinations of $\{1, x, x^2, \cdots, x^n\}$ form a subspace of the function space $C(-\infty, \infty)$.

(E). Let $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be a set of vectors in a vector space $\mathbf{V}$. If there is a nonzero triple of scalars (c_1, c_2, c_3) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

then the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

Problem 7. (2 points.) Determine which of the following statements is wrong.

(A). A set of two or more vectors is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

(B). A finite set in $\mathbb{R}^n$ that contains the zero vector $\mathbf{0}$ is linearly independent.

(C). Let $\mathbf{S} = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ be a set of vectors in $\mathbb{R}^n$ and $r > n$. Then $\mathbf{S}$ is linearly dependent.

(D). The set of vectors $\mathbf{v}_1 = (1, -2, 3)$, $\mathbf{v}_2 = (5, 6, -1)$ and $\mathbf{v}_3 = (3, 2, 1)$ is linearly dependent.

(E). If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ is linearly independent, then the set $\{\mathbf{v_1} + \mathbf{v_2}, \mathbf{v_2} + \mathbf{v_3}, \mathbf{v_3} + \mathbf{v_1}\}$ is linearly independent.

Problem 8. (2 points.) Determine which of the following statements is true.

(A). If $\mathbf{V} = \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$, then $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_r}\}$ is a basis for $\mathbf{V}$.

(B). Every linearly independent set of a vector space is a basis for $\mathbf{V}$.

(C). In $\mathbb{R}^2$, $\mathbf{u}_1 = (1, 1)$ and $\mathbf{u}_2 = (-1, 1)$ is a basis for $\mathbb{R}^2$.

(D). Let $\mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (2, 2, 0)$ and $\mathbf{v}_3 = (3, 3, 3)$ be a basis for $\mathbb{R}^3$. Then the vector with coordinate vector $(1, 2, 3)$ relative to this basis is $(14, 10, 9)$.

(E). Let $p = x^2 - 3x + 4$. Then the coordinate vector of p with respect to the basis $\{1, -x, x^2\}$ in $\mathbf{P}_2$ is $(4, -3, 1)$.

Problem 9. **(a). (2 point.)** Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a set in $\mathbb{R}^3$ where

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = (1,2,1), \quad \mathbf{v}_3 = (1,0,2).$$

Show that S is a basis for $\mathbb{R}^3$.

Proof. **(1).** We prove that S is linearly independent in $\mathbb{R}^3$. That is to say,

$$a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = (0,0,0)$$

implies that $a = b = c = 0$. We write it in the linear system

$$\begin{cases} a+b+c &= 0, \\ a+2b &= 0, \\ a+b+c &= 0. \end{cases}$$

We apply the Gauss-Jordan elimination to reduce

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus $a = 0, b = 0, c = 0$, which proves that S is linearly independent.

(2). We prove that S spans $\mathbb{R}^3$. That is for any (b_1, b_2, b_3) in $\mathbb{R}^3$, there exists $(a, b.c)$ such that

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = (b_1, b_2, b_3),$$

i.e.,

$$\begin{cases} a+b+c &= b_1, \\ a+2b &= b_2, \\ a+b+c &= b_3. \end{cases}$$

By part **(1)**, the coefficient matrix can be reduced to the identity matrix and so is invertible. Thus there exists a solution to this linear system.

To conclude, S is a basis.

□

(b). (1 point.) Find the coordinate vector (a, b, c) of $\mathbf{v} = (1, 0, 0)$.

Proof.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Thus the coordinate vector is $(4, -2, -1)$.

□

(c). (1 point.) Find the vector $\mathbf{v}$ in $\mathbb{R}^3$ whose coordinate vector relative to S is $(\mathbf{v})_S = (3, 2, 0)$.

Proof.

$$3\mathbf{v}_1 + 2\mathbf{v}_2 = 3(1, 1, 1) + 2(1, 2, 1) = (5, 7, 5).$$

□