SOME PROBLEMS FOR MATH 890

Question. 1. Let $f, g \in L^2(\mathbb{R}^n)$. Show that the multiplication formula holds:

$$\int \hat{f} g = \int f \hat{g}.$$

Question 2. The Hausdorff-Young inequality asserts that

$$\|\hat{f}\|_{p'} \leq A_p \|f\|_p, 1 \leq p \leq 2,$$

where p' is conjugate to p, $\frac{1}{p'} + \frac{1}{p} = 1$. The following two exercises clarify that the conditions on p are necessary.

(i). Show that, if $\|\hat{f}\|_q \leq A_{p,q} \|f\|_p$, then $q = p'$.
(ii). Show that, if $\|\hat{f}\|_{p'} \leq A_p \|f\|_p$, then $1 \leq p \leq 2$.

Question 3. Let $T : L^2 \to L^2$ be defined by

$$Tf(x) = \int K(x, y) f(y) dy.$$

Suppose that there exists $0 < A, B < \infty$ such that

$$\int |K(x, y)| dy \leq A, \text{ for a.e. } x,$$

and

$$\int |K(x, y)| dx \leq B, \text{ for a.e. } y.$$

Then

$$\|Tf\|_2 \leq A^{1/2} B^{1/2} \|f\|_2.$$

Question 4. Prove that for $f, g \in L^p$, $1 < p < 2$,

$$\int \hat{f} g = \int f \hat{g}.$$

Question 5. (i). If $f \in L^1, g \in L^2$, prove that $\widehat{f * g} = \hat{f} \hat{g}$.

(ii). If $f \in L^1, g \in L^p, 1 \leq p \leq 2$, then

$$\widehat{f * g} = \hat{f}\hat{g}.$$

Question 6. Prove that if f is C_0, then f is uniformly continuous.

Question 7. Prove that if P is the Poisson integral, then for any locally integral function f,

$$f * P \leq CM(f)$$

where $C > 0$ is some constant, and M is the Hardy-Littlewood maximal function.

Question 8. Prove that if $f \in L^p$, $1 \leq p \leq \infty$, then if x_0 is a Lebesgue point of f,

$$\int_{\mathbb{R}^n} |f(x_0 - t) - f(x_0)| P(t, y) dt \to 0, \text{ as } y \to 0.$$

Question 9. Let $s \geq 0$ be a subharmonic harmonic function. Suppose s has continuous second derivatives in a domain $\mathcal{D}$. Prove that $\Delta s \geq 0$.

Question 10. Let s be a subharmonic function in $\mathbb{R}^+_{n+1}$ satisfying that

$$\int_{\mathbb{R}^n} |s(x, y)|^q dx \leq c^q < \infty$$

for some $c > 0$ and all $y > 0$. Given $\varepsilon > 0$, define

$$m_\varepsilon(x, y) = \int s(t, \varepsilon) p(x - t, y) dt$$

where p is the Poisson kernel. In class we prove that $s(x, y + \epsilon) \leq m_\varepsilon(x, y)$ for $(x, y) \in R^+_{n+1}$. Use the same proof to show that m_ε is the least harmonic majorant of $s(x, y + \epsilon)$.

Problem 11. Define

$$A_p := \sup_{f \neq 0} \frac{\|\hat{f}\|_{p'}}{\|f\|_p}, 1 \leq p \leq 2,$$

where p' is the conjugate exponent to p, and for $1 \leq p, q, r \leq \infty$,

$$B = \sup_{f, g \neq 0} \frac{\|f * g\|_r}{\|f\|_p \|g\|_q}, \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Show that if $1 \leq p, q, r' \leq 2$,

$$B \leq A_{r'} A_p A_q.$$

Problem 12. Let f be a function in $C_c^\infty(\mathbb{R}^n)$ and $0 < \alpha < n$. Then with $c_\alpha = \frac{\Gamma(\frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}}}$,

$$c_\alpha(|\xi|^{-\alpha}\hat{f}(\xi))^\vee = c_{n-\alpha}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Here $f^\vee$ denotes the inverse Fourier transform of f.

Problem 13. Establish the following identity, for any $y > 0$,

$$\int_0^\infty e^{2\pi i x t}e^{-2\pi y t}dt = -\frac{1}{2\pi i z},$$

where $z = x + iy$. Hint: use the the Fourier transforms related to the Poisson kernel and the conjugate Poisson kernel.

Problem 14. Let σ be the surface measure on the unit sphere S^{n-1} in $\mathbb{R}^n$. Prove that

$$\widehat{d\sigma}(\xi) = 2\pi|\xi|^{-\frac{n-2}{2}}J_{\frac{n-2}{2}}(2\pi|\xi|),$$

where $\widehat{d\sigma}(\xi) = \int_{S^{n-1}} e^{-2\pi i\xi\cdot x}d\sigma(x)$. Hint: use the proof to prove Theorem 3.3 in the class regarding the representation formula of the Fourier transform of radial functions.

Problem 15. For f smooth functions on the sphere S^2, we can similarly define $\widehat{f\sigma}$ as in **Problem 14**,

$$\widehat{f\sigma}(\xi) = \int_{S^{n-1}} e^{-2\pi i\xi\cdot x}f(x)d\sigma(x).$$

Suppose that

$$\|\widehat{f\sigma}(\xi)\|_q \le C_{p,q}\|f\|_{L^p(S^{n-1})}$$

for some constant $C_{p,q}$ independent of f. Then p, q satisfy the following conditions

$$q > 3, \frac{2}{q} \le \frac{1}{p'}.$$

Hint: for the first condition, use **Problem 14** and the asymptotic formula of the Bessel functions. For the second condition, exploit the curvature of the unit sphere. Establishing these conditions are sufficient is a major open problem in harmonic analysis, which is known as Stein's restriction conjecture.