HOMEWORK 10

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1. P221. Ex. 1

Proof. If $c \neq 0$, for any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{|c|}$, then for $|x-y| \leq \delta$,

$$|f(x) - f(y)| = |c| \times |x-y| \leq |c| \times \delta = \varepsilon.$$

If $c = 0$, $f \equiv 0$.

We conclude that f is uniformly continuous in both cases.

□

2. P221. Ex. 2

Proof. From $|f'| < K$, we see that $K > 0$. By the mean value theorem,

$$|f(x) - f(y)| = |f'(c)||x-y| < K|x-y|.$$

For any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{K}$, then for $|x-y| \leq \delta$,

$$|f(x) - f(y)| \leq K \times \delta = \varepsilon.$$

Then f is uniformly continuous. □

3. P221. Ex. 3

Proof. Given $\varepsilon > 0$. We choose $\delta = \varepsilon^2$.

If $x, y \in (\frac{\delta}{4}, \infty)$, then $\sqrt{x}, \sqrt{y} \geq \frac{\sqrt{\delta}}{2} = \frac{\varepsilon}{2}$,

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x-y|}{\varepsilon} \leq \frac{\delta}{2\varepsilon} = \frac{\varepsilon}{2} \leq \varepsilon.$$

If $x, y \in (0, \delta)$, then $\sqrt{x}, \sqrt{y} \leq \sqrt{\delta} = \varepsilon$, then

$$|\sqrt{x} - \sqrt{y}| \leq \max\{\sqrt{x}, \sqrt{y}\} \leq \varepsilon.$$

In general, if $x, y \in (0, \infty)$ and $|x-y| \leq \frac{\delta}{2}$, then either $x, y \in (0, \delta)$ or $(\frac{\delta}{4}, \infty)$. Hence in either case

$$|f(x) - f(y)| \leq \varepsilon.$$

Then $f(x) = \sqrt{x}$ is uniformly continuous on $(0, \infty)$.

There is another way to prove it. It starts with the following claim, for $0 < y < x$, we write $x = (x - y) + y$

$$\sqrt{x} \leq \sqrt{x-y} + \sqrt{y}, \Rightarrow \sqrt{x} - \sqrt{y} \leq \sqrt{x-y}.$$

So for $\varepsilon > 0$, there exists $\delta = \varepsilon^2$, such that for any $|x - y| \leq \delta$,

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|} \leq \varepsilon.$$

□

4. P221 Ex. 5

Proof. $\{x_n\}$ is a Cauchy sequence: for any $\delta > 0$, there exists $N \in \mathcal{N}$ such that

$$|x_{n+p} - x_n| \leq \delta, \text{ for any p } \in \mathcal{N}.$$

f is uniformly continuous: for any $\varepsilon > 0$, there exists $\delta_0 > 0$, for any $|x - y| \leq \delta_0$,

$$|f(x) - f(y)| \leq \varepsilon.$$

We take $\delta = \delta_0$, then

$$|f(x_{n+p}) - f(x_n)| \leq \varepsilon, \text{ for any p } \in \mathcal{N}.$$

This shows that $\{f(x_n)\}$ is also a Cauchy sequence. □

5. P 226 Ex. 2

Proof. The function we consider is $f(x) = x$. On the interval $I_k = [\frac{k-1}{n}, \frac{k}{n}]$ for $1 \leq k \leq n$,

$$\min_{x \in I_k} f(x) = \frac{k-1}{n}, \max_{x \in I_k} f(x) = \frac{k}{n}.$$

Then

$$L_P(x) = \sum_{k=1}^{n} \frac{k-1}{n} \times \frac{1}{n} = \frac{\sum_{k=1}^{n}(k-1)}{n^2} = \frac{n-1}{2n}.$$

On the other hand,

$$U_P(x) = \sum_{k=1}^{n} \frac{k}{n} \times \frac{1}{n} = \frac{\sum_{k=1}^{n} k}{n^2} = \frac{n+1}{2n}.$$

□

6. P 234 Ex. 3

Proof. An useful observation is the following: for any interval I,

$$\inf_{x\in I} f(x)+\inf_{x\in I} f(x) \leq \inf_{x\in I} f(x)+g(x) \leq \sup_{x\in I} f(x)+g(x) \leq \sup_{x\in I} f(x)+\sup_{x\in I} g(x).$$

Then for P denote any partition of $[a,b]$,

$$L_P(f)+L_P(g) \leq L_P(f+g) \leq U_P(f+g) \leq U_P(f)+U_Q(g).$$

Since both f and g are integrable, then there exists P_1 and P_2,

$$U_{P_1}(f)-\varepsilon \leq \int_a^b f(x)dx \leq L_{P_1}(f)+\varepsilon$$

and

$$U_{P_2}(g)-\varepsilon \leq \int_a^b f(x)dx \leq L_{P_2}(g)+\varepsilon.$$

Then taking the union P_1 and P_2,

$$U_{P_1\cup P_2}(f)+U_{P_1\cup P_2}(g)-2\varepsilon \leq \int_a^b f(x)dx+\int_a^b g(x)dx \leq L_{P_1\cup P_2}(f)+L_{P_1\cup P_2}(g)+2\varepsilon.$$

Then we have

$$(1) \quad U_{P_1\cup P_2}(f+g)-2\varepsilon \leq \int_a^b f(x)dx+\int_a^b g(x)dx \leq L_{P_1\cup P_2}(f+g)+2\varepsilon.$$

In particular, it implies that

$$|U_{P_1\cup P_2}(f+g)-L_{P_1\cup P_2}(f+g)| \leq 4\varepsilon.$$

Hence $f+g$ is integrable. Denote the integral by $\int_a^b (f+g)(x)dx$.

Since $L_{P_1\cup P_2}(f+g) \leq \int_a^b (f+g)dx \leq U_{P_1\cup P_2}(f+g)$, the inequality (1) also implies that

$$\left| \int_a^b f(x)+g(x)dx - \int_a^b f(x)dx - \int_a^b g(x)dx \right| \leq 2\varepsilon.$$

Since ε is arbitrary, we see that

$$\int_a^b f(x)+g(x)dx = \int_a^b f(x)dx + \int_a^b g(x).$$

$\square$

7. P 235 Ex. 5

Proof. We calculate it by using the fundamental theorem of Calculus. Since $\left(\frac{x^{n+1}}{n+1}\right)' = x^n$,

$$\int_a^b x^n dx = \frac{x^{n+1}}{n+1}|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

□

8. P 235 Ex. 6

Proof. We calculate it by using the fundamental theorem of Calculus. Since $(\sin x)' = \cos x$,

$$\int_a^b \cos x dx = \sin x|_a^b = \sin b - \sin a.$$

□

9. P 235 Ex. 7

Proof. We calculate it by using the fundamental theorem of Calculus. Since $(e^x)' = e^x$,

$$\int_a^b e^x dx = e^x|_a^b = e^b - e^a.$$

□

10. P 235 Ex. 8

Proof. Since $\ln x = (x \ln x)' - x(\ln x)' = (x \ln x)' - x\frac{1}{x} = (x \ln x)' - 1$ by the product rule, we see that

$$\int_a^b \ln x dx = \int_a^b ((x \ln x)' - 1)dx = b \ln b - a \ln a - (b - a).$$

□

11. P 235 Ex. 10

Proof. We prove it by contradiction. Suppose that $f(x_0) > 0$. Then by the continuity of f, there exists $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset [a, b]$ and

$$f(x) > \frac{f(x_0)}{2}, \text{ for } x \in [x_0 - \delta/2, x_0 + \delta/2].$$

We construct a continuous function g such that g is nonnegative and symmetric about $x = x_0$ satisfying that

$$g(x_0) = 1, g(x_0 \pm \frac{\delta}{2}) = \frac{1}{2}, \text{ and } g(x_0 \pm \delta) = 0,$$

and g is 0 outside of $[x_0 - \delta, x_0 + \delta]$. Then fg is continuous on $[a, b]$ and hence is integrable. Then

$$\int_a^b f(x)g(x)dx \geq \int_{x_0-\delta/2}^{x_0+\delta/2} f(x)g(x)dx \geq \frac{1}{2}\int_{x_0-\delta/2}^{x_0+\delta/2} f(x)dx \geq \frac{f(x_0)\delta}{4} > 0.$$

This contradicts to $\int_a^b f(x)g(x)dx = 0$. Hence $f \equiv 0$. □

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