

HOMEWORK 3

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ABSTRACT. Please send me an email if you find mistakes. Thanks.

1. P91. EXERCISE 7.1

For convenience of writing, we will replace n' by N in the reasoning below.

(b) $\{\frac{1}{2n+3}\} \rightarrow 0$.

Given any $\varepsilon > 0$. We need to find $N > 0$ such that for any $n \geq N$,

$$\frac{1}{2n+3} - 0 \leq \varepsilon.$$

That is to say,

$$\frac{1}{2n+3} \leq \varepsilon, \Rightarrow 2n \geq \frac{1}{\varepsilon} - 3, \Rightarrow n \geq \frac{1}{2\varepsilon} - \frac{3}{2}.$$

From this, we may take $N = [\frac{1}{2\varepsilon} - \frac{3}{2}] + 1$, where $[x]$ denotes the greatest integer less than or equal to x , for instance, $[3.5] = 3$, $[-1/2] = -1$. In the case of $[\frac{1}{2\varepsilon} - \frac{3}{2}] + 1 \leq 0$, we take $N = 1$. So we may take

$$N = \max \left\{ \left[\frac{1}{2\varepsilon} - \frac{3}{2} \right] + 1, 1 \right\}.$$

Now we formally write: for any $\varepsilon > 0$, there exists $N = \max \{ [\frac{1}{2\varepsilon} - \frac{3}{2}] + 1, 1 \}$ such that for any $n \geq N$,

$$\frac{1}{2n+3} - 0 \leq \varepsilon.$$

So

$$\lim_{n \rightarrow \infty} \frac{1}{2n+3} = 0.$$

(d) $\frac{7n-5}{5n+7} \rightarrow \frac{7}{5}$.

As above, given any $\varepsilon > 0$. We need to find $N > 0$ such that for any $n \geq N$,

$$\frac{7n-5}{5n+7} - \frac{7}{5} \leq \varepsilon.$$

That is to say,

$$\frac{74}{5(5n+7)} \leq \varepsilon.$$

So we may take $N = \left[\frac{74}{25\varepsilon} - \frac{7}{5} \right] + 1$ in the case where $\left[\frac{74}{25\varepsilon} - \frac{7}{5} \right] + 1 \geq 1$; otherwise we take $N = 1$.

We formally write: for any $\varepsilon > 0$, there exists $N = \max \left\{ \left[\frac{74}{25\varepsilon} - \frac{7}{5} \right] + 1, 1 \right\}$ such that for any $n \geq N$,

$$\frac{7n-5}{5n+7} - \frac{7}{5} \leq \varepsilon.$$

So

$$\lim_{n \rightarrow \infty} \frac{7n-5}{5n+7} = \frac{7}{5}.$$

(f). Given $\varepsilon > 0$. We need to find $N > 0$ such that for any $n \geq N$,

$$\frac{2n^2-1}{3n^2+2} - \frac{2}{3} \leq \varepsilon.$$

That is to say,

$$n^2 \geq \frac{7}{9\varepsilon} - \frac{2}{3}.$$

When $\frac{7}{9\varepsilon} - \frac{2}{3} \leq 0$, we take $N = 1$; otherwise $N = \left[\sqrt{\frac{7}{9\varepsilon} - \frac{2}{3}} \right] + 1$.

We formally write: for any $\varepsilon > 0$, there exists $N = \max \left\{ 1, \left[\sqrt{\frac{7}{9\varepsilon} - \frac{2}{3}} \right] + 1 \right\}$ such that for any $n \geq N$,

$$\frac{2n^2-1}{3n^2+2} - \frac{2}{3} \leq \varepsilon.$$

So

$$\lim_{n \rightarrow \infty} \frac{2n^2-1}{3n^2+2} = \frac{2}{3}.$$

(h). Given $\varepsilon > 0$. We need to find $N > 0$ such that for any $n \geq N$,

$$\frac{n}{n^2-1} - 0 \leq \varepsilon.$$

For $n \geq 2$, $\frac{n}{n^2-1} > 0$. Then $\frac{n}{\varepsilon} \leq n^2 - 1 \leq n^2$. Thus

$$\frac{n}{\varepsilon} \leq n^2, \Rightarrow n \geq \frac{1}{\varepsilon}.$$

So we may take $N = \left[\frac{1}{\varepsilon} \right] + 1$.

We write: for any $\varepsilon > 0$, there exists $N = \left[\frac{1}{\varepsilon}\right] + 1$ such that for any $n \geq N$,

$$\frac{n}{n^2 - 1} - 0 \leq \varepsilon.$$

So

$$\lim_{n \rightarrow \infty} \frac{n}{n^2 - 1} = 0.$$

(j). Given any $\varepsilon > 0$, we need to find $N > 0$ such that for any $n \geq N$,

$$\frac{n}{n^2 - n + 1} - 0 \leq \varepsilon.$$

This implies that

$$\frac{n}{\varepsilon} \leq n^2 - n + 1 \leq n^2 + 1 \leq 2n^2.$$

So $n \geq \frac{1}{2\varepsilon}$. We may take $N = \left[\frac{1}{2\varepsilon}\right] + 1$.

We write: for any $\varepsilon > 0$, there exists $N = \left[\frac{1}{2\varepsilon}\right] + 1$ such that for any $n \geq N$,

$$\frac{n}{n^2 - n + 1} - 0 \leq \varepsilon.$$

Hence

$$\lim_{n \rightarrow \infty} \frac{n}{n^2 - n + 1} = 0.$$

(l). For any $\varepsilon > 0$, we need to find $N > 0$ such that for any $n \geq N$,

$$\frac{\sqrt{n} + 1}{3\sqrt{n} - 30} - \frac{1}{3} \leq \varepsilon.$$

That is to say

$$\frac{11}{3\sqrt{n} - 30} \leq \varepsilon.$$

When $n \geq 200$, $\frac{11}{3\sqrt{n} - 30} > 0$. So

$$\sqrt{n} \geq \frac{11}{3\varepsilon} + 10, \Rightarrow n \geq \left(\frac{11}{3\varepsilon} + 10\right)^2.$$

We may take

$$N = \left[\left(\frac{11}{3\varepsilon} + 10\right)^2\right] + 1.$$

We formally write: for any $\varepsilon > 0$, there exists $N = \left[\left(\frac{11}{3\varepsilon} + 10\right)^2\right] + 1$ such that

$$\frac{\sqrt{n} + 1}{3\sqrt{n} - 30} - \frac{1}{3} \leq \varepsilon.$$

So

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n} + 1}{3\sqrt{n} - 30} = \frac{1}{3}.$$

2. P 91. Ex. 2.

Proof. We may assume that $A > 0$. $\lim_{n \rightarrow \infty} a_n = A$. Then for $\varepsilon > 0$ but $\varepsilon < \frac{A}{2}$, there exists $N > 0$ such that, for any $n \geq N$,

$$|a_n - A| \leq \varepsilon, \Rightarrow A - \varepsilon \leq a_n \leq A + \varepsilon.$$

Fixing N . Then for $n > N$,

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{a_1 + a_2 + \cdots + a_N}{n} + \frac{a_{N+1} + a_{N+2} + \cdots + a_n}{n}.$$

Since $a_{N+j} \geq A - \varepsilon$ for $1 \leq j \leq n - N$,

$$\frac{a_{N+1} + a_{N+2} + \cdots + a_n}{n} \geq \frac{(n - N)(A - \varepsilon)}{n} = \left(1 - \frac{N}{n}\right)(A - \varepsilon).$$

On the other hand, since $a_{N+j} \leq A + \varepsilon$ for $1 \leq j \leq n - N$,

$$\frac{a_{N+1} + a_{N+2} + \cdots + a_n}{n} \leq \frac{(n - N)(A + \varepsilon)}{n} = \left(1 - \frac{N}{n}\right)(A + \varepsilon).$$

Then we see that

$$\frac{a_1 + a_2 + \cdots + a_N}{n} + \left(1 - \frac{N}{n}\right)(A - \varepsilon) \leq b_n \leq \frac{a_1 + a_2 + \cdots + a_N}{n} + \left(1 - \frac{N}{n}\right)(A + \varepsilon).$$

Since $\lim_{n \rightarrow \infty} \frac{a}{n} = 0$ for any real number a ,

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \cdots + a_N}{n} - \frac{N(A - \varepsilon)}{n} = 0, \text{ or } \lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \cdots + a_N}{n} - \frac{N(A + \varepsilon)}{n} = 0.$$

Then for the same $\varepsilon > 0$,

$$\frac{a_1 + a_2 + \cdots + a_N}{n} - \frac{N(A - \varepsilon)}{n} < \varepsilon, \quad \frac{a_1 + a_2 + \cdots + a_N}{n} - \frac{N(A + \varepsilon)}{n} < \varepsilon.$$

In particular,

$$\frac{a_1 + a_2 + \cdots + a_N}{n} - \frac{N(A - \varepsilon)}{n} \geq \varepsilon, \quad \frac{a_1 + a_2 + \cdots + a_N}{n} - \frac{N(A + \varepsilon)}{n} \leq \varepsilon.$$

So

$$A - 2\varepsilon \leq b_n \leq A + 2\varepsilon.$$

Thus every interval of A contains almost all terms of $\{b_n\}$. Hence

$$\lim_{n \rightarrow \infty} b_n = A.$$

□

3. P 92, Ex. 4.

Proof. Since $a_n \rightarrow A$, then for any $\varepsilon > 0$, there exists $N_1 > 0$ such that for $n \geq N_1$,

$$|a_n - A| \leq \varepsilon.$$

Since $a_n \rightarrow B$, then for any $\varepsilon > 0$, there exists $N_2 > 0$ such that for $n \geq N_2$,

$$|a_n - B| \leq \varepsilon.$$

Then for $n \geq \max\{N_1, N_2\}$,

$$|A - B| = |a_n - A| + |a_n - B| \leq 2\varepsilon.$$

But 2ε can be arbitrarily small; that is to say, $|A - B|$ can be arbitrarily small. Then $|A - B| = 0$, which implies that $A = B$. \square

4. P 92, Ex. 6.

Proof. It is already known that A is an upper bound of S . Then for any $x \in S$, $x \leq A$. In order to establish the claim, we first show one direction.

\Leftarrow . In view of the above, we just need to show that A is the least one among the upper bounds of S . Given any upper bound b of S .

We assume that (a) holds: $A \in S$. Then $A \leq b$, which says that A is the least upper bound.

We assume that (b) holds. A is the limit of some sequence $\{a_n\} \in S$. Since $a_n \in S$, then $a_n \leq b$ for all n . Then

$$\lim_{n \rightarrow \infty} a_n \leq b, \Rightarrow A \leq b.$$

This also implies that A is the least upper bound.

\Rightarrow . Assume that A is the least upper bound of S . There are two possibilities: either $A \in S$ or A is not in S . If $A \in S$, then that is the case (a). If not, for any positive integer $n \in \mathbb{N}$, $A - \frac{1}{n}$ is not an upper bound of S because A is the least upper bound of S . That is to say, there exists $a_n \in S$ such that

$$a_n > A - \frac{1}{n}.$$

So

$$A - \frac{1}{n} < a_n \leq A \leq A + \frac{1}{n}.$$

This says that every interval of A contains almost all terms of $\{a_n\}$. Hence

$$\lim_{n \rightarrow \infty} a_n = A.$$

This establishes (b). \square

5. P 99. Ex. 1

(b).

$$\lim_{n \rightarrow \infty} \frac{3n^2 + 5n + 1}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{3 + \frac{5}{n} + \frac{1}{n^2}}{1 + \frac{1}{n^2}} = \frac{3}{1} = 3.$$

(d).

$$\lim_{n \rightarrow \infty} \frac{(1/2)^n + 3 \sin n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} (1/2)^n + \frac{3}{\sqrt{n}} \sin n = 0 + \lim_{n \rightarrow \infty} \frac{3}{\sqrt{n}} \sin n = 0$$

as

$$\frac{3}{\sqrt{n}} \sin n \leq \frac{3}{\sqrt{n}}.$$

(f).

$$\lim_{n \rightarrow \infty} \frac{2^n + 1}{3^n - 1} = \lim_{n \rightarrow \infty} \frac{(2/3)^n + 3^{-n}}{1 - 3^{-n}} = \frac{0 + 0}{1 - 0} = 0$$

as $2/3 < 1$ and $1/3 < 1$.

(h). By the same reasoning as (f), the limit is zero too.

(j). The sequence diverges. For $n \geq 2$,

$$n^n \geq 2^n, \lim_{n \rightarrow \infty} 2^n = \infty.$$

(l). $\frac{\sin n}{n} \leq \frac{1}{n}$. Then

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\sin n}{n} \right) = 1.$$

(n). We know that

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} \geq \frac{n(n-1)(n-2)}{6}.$$

Then we see that

$$\frac{n^2}{2^n} \leq \frac{6n^2}{n(n-1)(n-2)} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus

$$\lim_{n \rightarrow \infty} \frac{n^2}{2^n} = 0.$$

(p).

$$\frac{n!}{2^n} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \cdots \times (n-1) \times n}{2 \times 2 \times \cdots \times 2} = \frac{1}{2} \times \frac{2}{2} \times \frac{3}{2} \times \frac{4}{2} \times \frac{5}{2} \cdots \times \frac{n}{2}.$$

Since $\frac{k}{2} \geq 2$ for $k \geq 4$, we see the above is less than

$$\frac{1}{2} \times \frac{3}{2} \times 2^{n-3} = \frac{3}{4} \times 2^{n-3}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{n!}{2^n} = 0.$$

6. P99. Ex. 3.

Proof.

$$\sqrt{n+a_n} - \sqrt{n} = \frac{a_n}{\sqrt{n+a_n} + \sqrt{n}} \rightarrow 0,$$

as $n \rightarrow \infty$. □

7. P99. Ex. 5.

Proof. We assume that $\{\frac{1}{a_n}\}$ converges to A . Then since $1 = a_n \times \frac{1}{a_n}$, by the limit theorem, we see that

$$1 = \lim_{n \rightarrow \infty} a_n \times \lim_{n \rightarrow \infty} \frac{1}{a_n} = 0 \times A = 0,$$

which is a contradiction. So $\{\frac{1}{a_n}\}$ diverges. □

8. P99, Ex. 7

This follows directly from the limit theorem.

9. P100. Ex. 10.

Proof. $\{b_n\}$ diverges.

$\{a_n\} \rightarrow 1$. Then for all large n , $a_n \geq \frac{1}{2}$. Then

$$b_n = na_n + n^2a_{n+1} \geq \frac{n}{2} + \frac{n^2}{4} \text{ but } \frac{n}{2} + \frac{n^2}{4} \rightarrow \infty.$$

This shows that

$$\lim_{n \rightarrow \infty} b_n = \infty.$$

□

10. P100. Ex. 12.

Proof. We begin the case where $p = 1$. Then we have $\{a_n\} \rightarrow A$ and ask what is the limit of

$$\frac{a_0 + a_1 + \cdots + a_n}{n}.$$

The limit is A by Ex. 2 on page 91.

When $p > 1$, the limit is 0 as

$$\frac{\sum_{k=0}^n a_k}{n^p} = \frac{1}{n^{p-1}} \frac{\sum_{k=0}^n a_k}{n} \rightarrow 0 \times A = 0.$$

When $p < 1$, the sequence diverges. We may assume that $A > 0$.

$$\frac{\sum_{k=0}^n a_k}{n^p} = n^{1-p} \frac{\sum_{k=0}^n a_k}{n} \geq n^{1-p} \times \frac{A}{2}.$$

The sequence $n^{1-p} \frac{A}{2}$ diverges. So the original sequence diverges, too. \square

11. P100. Ex. 13.

Proof. We assume that $a_n = (-1)^n$. Then

$$\sum_{k=0}^n a_k = 0, \text{ when } n \text{ is even, } = -1 \text{ when } n \text{ is odd.}$$

So the sequence

$$\frac{1}{n^p} \sum_{k=0}^n a_k \leq \frac{1}{n^p} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

So

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} \sum_{k=0}^n a_k = 0.$$

\square

12. P 101. Ex. 25.

Proof. We first observe that $a_n \geq 1$. We assume that $a_n = 1 + \alpha_n$ with $\alpha_n \geq 0$ for all n . Then

$$a_n^n = (1 + \alpha_n)^n = n.$$

By the Binomial Theorem,

$$(1 + \alpha_n)^n = 1 + n\alpha_n + \binom{n}{2} \alpha_n^2 + \cdots + \alpha_n^n.$$

Every term in the summation is non-negative, then

$$\binom{n}{2} \alpha_n^2 \leq n \Rightarrow \alpha_n \leq \frac{2}{n-1}.$$

So $\alpha_n \rightarrow 0$, which in turn implies that $a_n \rightarrow 1$. □

13. P 101. Ex. 27.

Proof. We first follow the hint. Since $0 < a < 1$, there exists $0 < h$ such that $a = \frac{1}{1+h}$. Then

$$na^n = \frac{n}{(1+h)^n} \leq \frac{n}{\binom{n}{2}h^2} = \frac{2n}{n(n-1)h^2} \rightarrow 0$$

as $n \rightarrow \infty$. This shows that $\lim_{n \rightarrow \infty} na^n = 0$.

Secondly we can also use the result in Ex. 25: $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$. Let $a_n = \sqrt[n]{na}$. Then

$$0 < a_n \rightarrow a < 1.$$

So for all sufficiently large n ,

$$0 < a_n < a + \delta < 1, \text{ for some } \delta > 0.$$

Then

$$0 < a_n^n = na^n \leq (a + \delta)^n, a + \delta < 1.$$

Then

$$\lim_{n \rightarrow \infty} na^n = 0.$$

□

14. P 101. Ex. 28.

Proof. We still use Ex. 25. Since $\sqrt[n]{n} = 1$, then for any fixed integer k ,

$$\sqrt[n]{n^k} = (\sqrt[n]{n})^k \rightarrow 1^k = 1,$$

as $n \rightarrow \infty$. If letting $a_n = \sqrt[n]{n^k}a$, then $a_n \rightarrow a$ as $n \rightarrow \infty$. Then the rest proof follows similarly.

We can also use the Binomial theorem. Choose $n > 2k$. Let $a = \frac{1}{1+h}$. Then

$$n^k a^n = \frac{n^k}{(1+h)^n} \leq \frac{n}{\binom{n}{k+1}h^{k+1}} = \frac{(k+1)!}{\frac{n(n-1)\cdots(n-k+1)}{n^k} \times (n-k)h^{k+1}}$$

which goes to zero as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} n^k a^n = 0.$$

□

15. P 101. Ex. 35.

Proof. Since the relation $a_{n+1} = \frac{a_n - 1}{2}$ holds for all n , then

$$\begin{aligned} a_{n+1} &= \frac{a_n}{2} - \frac{1}{2} = \frac{a_{n-1}}{2^2} - \frac{1}{2^2} - \frac{1}{2} = \frac{a_{n-2}}{2^3} - \frac{1}{2^3} - \frac{1}{2^2} - \frac{1}{2} \\ &= \frac{a_0}{2^{n+1}} - \frac{1}{2^{n+1}} - \frac{1}{2^n} - \cdots - \frac{1}{2}. \\ &= \frac{a_0}{2^{n+1}} - \left(1 - \frac{1}{2^{n+1}}\right). \end{aligned}$$

So

$$\lim_{n \rightarrow \infty} a_n = -1.$$

□

16. P 101. Ex. 35.

Proof. The proof follows similarly as in Ex. 35. We omit it.

□

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