HOMEWORK 4

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1. P108. Ex 1

Proof. **1.** The sequence diverges. If the sequence converges, assume that $a_n \to A$ for some $A \in \mathbb{R}$. Then taking limits on both sides of $a_{n+1} = \sqrt{1+a_n^2}$, we see that

$$A = \sqrt{1+A^2}, \Rightarrow A^2 = 1 + A^2.$$

This is absurd. So the sequence diverges.

4. The sequence converges because it is monotone and bounded. Firstly $a_0 = 1 \geq 0$ and $a_{n+1} = \sqrt{3+2a_n} \geq 0$ because it is a square root of some number. On the other hand,

$$a_{n+1} - a_n = \sqrt{3+2a_n} - \sqrt{3+2a_{n-1}} = \frac{2(a_n - a_{n-1})}{\sqrt{3+2a_n} + \sqrt{3+2a_{n-1}}}.$$

So $a_{n+1} - a_n$ has the same sign as the previous difference $a_n - a_{n-1}$; this applies to all $n \geq 1$. Since $a_1 - a_0 = \sqrt{3+2a_0} - a_0 = \sqrt{5} - 1 \geq 0$,

$$a_{n+1} - a_n \geq 0.$$

So $\{a_n\}$ is increasing.

We claim that $a_n \leq 3$ for all n and prove it by mathematical induction. Firstly $a_0 = 1 \leq 3$. Then assume that $a_n \leq 3$,

$$a_{n+1} = \sqrt{3+2a_n} \leq \sqrt{3+2\times 3} \leq 3.$$

So $a_n \leq 3$ for all n.

We take limits on both sides of $a_{n+1} = \sqrt{3+2a_n}$, then

$$A = \sqrt{3+2A}, \Rightarrow A^2 - 2A - 3 = 0.$$

So $A = 3$ and $A = -2$. Since $a_n \geq 0$, then $A = 3$.

5. The sequence is convergent because it is monotone and bounded.

We first show that it is bounded. Since $a_0 = 1$ and $a_{n+1} = \sqrt{a_n + 1/a_n}$, then $a_n > 0$ for all n. On the other hand,

$$a_n + \frac{1}{a_n} \geq \sqrt{2a_n \frac{1}{a_n}} = \sqrt{2}.$$

Then $a_{n+1} \geq \sqrt{2}$ for all $n \geq 0$. We then prove by mathematical induction that $a_n \leq 2$. It is known that $a_0 \leq 2$. Then we assume that $a_n \leq 2$.

$$a_{n+1} = \sqrt{a_n + \frac{1}{a_n}} \leq \sqrt{2 + \frac{1}{\sqrt{2}}} \leq \sqrt{4} = 2.$$

So we conclude that $a_n \leq 2$.

Next we show that a_n is increasing.

$$\begin{aligned} a_{n+1} - a_n &= \sqrt{a_n + \frac{1}{a_n}} - \sqrt{a_{n-1} + \frac{1}{a_{n-1}}} \\ &= \frac{a_n + \frac{1}{a_n} - a_{n-1} - \frac{1}{a_{n-1}}}{\sqrt{a_n + \frac{1}{a_n}} + \sqrt{a_{n-1} + \frac{1}{a_{n-1}}}} \\ &= \frac{(a_n - a_{n-1})(1 - \frac{1}{a_n a_{n-1}})}{\sqrt{a_n + \frac{1}{a_n}} + \sqrt{a_{n-1} + \frac{1}{a_{n-1}}}} \end{aligned}$$

Since $a_n \geq \sqrt{2} \geq 1$, so $1 - \frac{1}{a_n a_{n-1}} > 0$. The denominator is already positive. So the sign of $a_{n+1} - a_n$ is the same as $a_n - a_{n-1}$. So

$$a_{n+1} \geq a_n$$

because $a_1 - a_0 = \sqrt{2} - 1 > 0$. That is $\{a_n\}$ is increasing.

So the limit of $\{a_n\}$ exists. Taking limits on both sides of $a_{n+1} = \sqrt{1 + \frac{1}{a_n}}$, we see that

$$A = \sqrt{A + \frac{1}{A}} \Rightarrow A^3 - A^2 - 1 = 0.$$

Solving it by calculators, $A \sim 1.46557$. □

2. P109. Ex. 15

Proof. The sequence is $a_0 = 1$ and $a_{n+1} = \sqrt{1 - a_n}$ for $n = 0, 1, 2, \cdots$. Then the sequence is

$$a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 0, a_4 = 1, \cdots ,$$

Thus the sequence is oscillating between 0 and 1. Thus the sequence is divergent. □

3. P 113 Ex. 1

Proof. Since $|a_{n+1} - a_n| < 2^{-n}$ for all $n > 0$, then for any $N \in \mathbb{N}$, and $m \in \mathbb{N}$,

$$\begin{aligned}
&|a_{N+1} - a_N| \leq 2^{-N},\\
&|a_{N+2} - a_{N+1}| \leq 2^{-N+1},\\
&|a_{N+3} - a_{N+2}| \leq 2^{-N+2},\\
&\quad \cdots\cdots\\
&|a_{N+m} - a_{N+m-1}| \leq 2^{-(N+m-1)}.
\end{aligned}$$

Then

$$|a_{N+m} - a_N| \leq 2^{-N} + \cdots + 2^{-(N+m-1)} = 2^N(1 + \frac{1}{2} + \cdots + \frac{1}{2^{-(m-1)}}) \leq 2^{-N} \times 1 = 2^{-N},$$

which goes to zero as N goes to infinity. This proves that $\{a_n\}$ is a Cauchy sequence, and thus is convergent. □

4. P 114. Ex. 2

Proof. It is known that

$$a_n = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots + \frac{(-1)^{n-1}}{n^n}.$$

Then for any $n, m \in \mathbb{N}$

$$\begin{aligned}
|a_{n+m} - a_n| &= \frac{(-1)^n}{(n+1)^{n+1}} + \frac{(-1)^{n+1}}{(n+2)^{n+2}} + \cdots + \frac{(-1)^{n+m-1}}{(n+m)^{n+m}}\\
&= \frac{1}{(n+1)^{n+1}} - \frac{1}{(n+2)^{n+2}} + \cdots + \frac{(-1)^{n+m-1}}{(n+m)^{n+m}}
\end{aligned} \tag{1}$$

We split it into two cases:

Case 1. If m is even, then we can group the terms in (1) into $m/2$-pairs,

$$\frac{1}{(n+1)^{n+1}} - \frac{1}{(n+2)^{n+2}}, \frac{1}{(n+3)^{n+3}} - \frac{1}{(n+4)^{n+4}}, \cdots, \frac{1}{(n+m-1)^{n+m-1}} - \frac{1}{(n+m)^{n+m}}$$

All are positive. Then

$$\begin{aligned}
|a_{n+m} - a_n| = &\frac{1}{(n+1)^{n+1}} - \frac{1}{(n+2)^{n+2}} + \cdots + \frac{(-1)^{n+m-1}}{(n+m)^{n+m}}\\
= &\left(\frac{1}{(n+1)^{n+1}} - \frac{1}{(n+2)^{n+2}}\right) + \left(\frac{1}{(n+3)^{n+3}} - \frac{1}{(n+4)^{n+4}}\right)\\
&+ \cdots + \left(\frac{1}{(n+m-1)^{n+m-1}} - \frac{1}{(n+m)^{n+m}}\right).
\end{aligned}$$

In this sum, we leave the first term and the last term, and group the middle terms into $\dfrac{m-2}{2}$-pairs,

$$\frac{1}{(n+2)^{n+2}} - \frac{1}{(n+3)^{n+3}}, \dots, \frac{1}{(n+m-2)^{n+m-2}} - \frac{1}{(n+m-1)^{n+m-1}}.$$

Then

$$\begin{aligned} |a_{n+m} - a_n| &= \frac{1}{(n+1)^{n+1}} - \left(\frac{1}{(n+2)^{n+2}} - \frac{1}{(n+3)^{n+3}}\right) \\ &\quad - \dots - \left(\frac{1}{(n+m-2)^{n+m-2}} - \frac{1}{(n+m-1)^{n+m-1}}\right) - \frac{1}{(n+m)^{n+m}} \\ &\leq \frac{1}{(n+1)^{n+1}}, \end{aligned}$$

which goes to zero as n goes to infinity. In this case, $\{a_n\}$ is a Cauchy sequence.

Case 2. If m is odd, then we group the first $m-1$ terms into $\dfrac{m-1}{2}$-pairs and change the sign of the last term into positive. Then

$$\begin{aligned} |a_{n+m} - a_n| &= \frac{1}{(n+1)^{n+1}} - \frac{1}{(n+2)^{n+2}} + \dots + \frac{(-1)^{n+m-1}}{(n+m)^{n+m}} \\ &\leq \left(\frac{1}{(n+1)^{n+1}} - \frac{1}{(n+2)^{n+2}}\right) + \left(\frac{1}{(n+3)^{n+3}} - \frac{1}{(n+4)^{n+4}}\right) \\ &\qquad + \dots + \left(\frac{1}{(n+m-2)^{n+m-2}} - \frac{1}{(n+m-1)^{n+m-1}}\right) + \frac{1}{(n+m)^{n+m}}. \end{aligned}$$

Then we leave the first term and group the rest $m-1$ into $\frac{m-1}{2}$-pairs, then

$$\frac{1}{(n+1)^{n+1}} - \left(\frac{1}{(n+2)^{n+2}} - \frac{1}{(n+3)^{n+3}}\right) - \dots - \left(\frac{1}{(n+m-1)^{n+m-1}} - \frac{1}{(n+m)^{n+m}}\right)$$

which is less than $\frac{1}{(n+1)^{n+1}}$. It goes to zero as n goes to infinity. So in this case, $\{a_n\}$ is also a Cauchy-sequence.

In both cases, $\{a_n\}$ converges. □

5. P114. Ex. 7.

Proof. It is known that $0 < c < 1$ and

$$|a_{n+1} - a_n| \leq c|a_n - a_{n-1}|.$$

Then for any $n, m \in \mathbb{N}$,

$$
\begin{aligned}
&|a_{n+2} - a_{n+1}| \leq c|a_{n+1} - a_n| \leq c^2|a_n - a_{n-1}| \\
&|a_{n+3} - a_{n+2}| \leq c|a_{n+2} - a_{n+1}| \leq c^3|a_n - a_{n-1}| \\
&\quad \cdots\cdots \\
&|a_{n+m} - a_{n+m-1}| \leq c|a_{n+m-1} - a_{n+m-2}| \leq c^m|a_n - a_{n-1}|
\end{aligned}
$$

Thus

$$|a_{n+m} - a_n| \leq (c + c^2 + \cdots + c^m)|a_n - a_{n-1}| \leq \frac{c}{1-c}|a_n - a_{n-1}|.$$

On the other hand, if we continue the process above,

$$|a_n - a_{n-1}| \leq c^{n-1}|a_1 - a_0|$$

Thus

$$|a_{n+m} - a_n| \leq \frac{c^n}{1-c}|a_1 - a_0|$$

which goes to zero as n goes to infinity.

This shows that $\{a_n\}$ is a Cauchy sequence. □

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