

HOMEWORK 5

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1. P119. Ex. 1

(b).

$$\sum \left(-\frac{3}{4}\right)^n.$$

This series converges as $|\frac{3}{4}| < 1$ and in fact

$$\sum \left(-\frac{3}{4}\right)^n = \frac{1}{1 - (-3/4)} = \frac{1}{7/4} = \frac{4}{7}.$$

Note that the series begins with $a_0 = (-3/4)^0 = 1$.

(f). The series $\sum (5(\frac{1}{2})^n + 12(\frac{1}{3})^n)$ converges as each of the terms $5\sum(\frac{1}{2})^n$ and $12\sum(\frac{1}{3})^n$ converges:

$$5\sum\left(\frac{1}{2}\right)^n = 5 \times \frac{1}{1 - (-\frac{2}{3})} = 3; 12\sum\left(\frac{3}{4}\right)^n = 12 \times \frac{1}{1 - \frac{3}{4}} = 12 \times 4 = 48.$$

So

$$\sum \left(5\left(\frac{1}{2}\right)^n + 12\left(\frac{1}{3}\right)^n\right) = 3 + 48 = 51.$$

2. P119&P120 Ex. 2

b). The series $\sum n^{-3/2}$ converges as $3/2 > 1$.

d). By Proposition 9.1.3, the series diverges as $a_n = 5n^{-3/4} + 2n^{3/2}$ does not converge to zero as n goes to infinity.

f). The series converges because both $\sum \frac{5}{2^n}$ and $\sum \frac{4}{3^n}$ converge.

h). By Proposition 9.1.3, the series diverges as $a_n = (-1)^n$ does not converge to zero as n goes to infinity.

3. P120. Ex. 7

Proof. For $x \in [n-1, n]$,

$$n(\ln n)^p \geq x(\ln x)^p, \Rightarrow \frac{1}{n(\ln n)^p} \leq \frac{1}{x(\ln x)^p}.$$

Then

$$\sum_{n \geq 3} \frac{1}{n(\ln n)^p} \leq \int_2^{\infty} \frac{1}{x(\ln x)^p} dx.$$

Evaluating this integral requires some calculus: for $p > 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\infty} \frac{1}{(\ln x)^p} d \ln x = \int_{\ln 2}^{\infty} \frac{1}{t^p} dt = \frac{(\ln 2)^{-p+1}}{p-1} < \infty.$$

So

$$\sum_{n \geq 2} \frac{1}{n(\ln n)^p} = \frac{1}{2(\ln 2)^p} + \sum_{n \geq 3} \frac{1}{n(\ln n)^p} \leq \frac{1}{2(\ln 2)^p} + \frac{(\ln 2)^{-p+1}}{p-1}.$$

In this case, it converges.

When $p \leq 1$, for $x \in [n, n+1]$,

$$n(\ln n)^p \leq x(\ln x)^p, \Rightarrow \frac{1}{n(\ln n)^p} \geq \frac{1}{x(\ln x)^p}.$$

Then

$$\sum_{n \geq 2} \frac{1}{n(\ln n)^p} \leq \int_2^{\infty} \frac{1}{x(\ln x)^p} dx.$$

Evaluating this integral requires some calculus: for $p = 1$ and any $M > 0$,

$$\int_2^M \frac{1}{x(\ln x)} dx = \int_{\ln 2}^{\ln M} \frac{1}{(\ln x)} d \ln x = \int_{\ln 2}^{\ln M} \frac{1}{t} dt = \ln \ln M - \ln \ln 2$$

which will exceed any finite number as M goes to infinity. So $\sum \frac{1}{n(\ln n)}$ diverges.

for $p < 1$,

$$\int_2^M \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\ln M} \frac{1}{(\ln x)^p} d \ln x = \int_{\ln 2}^{\ln M} \frac{1}{t^p} dt = \frac{1}{1-p} t^{1-p} \Big|_{\ln 2}^{\ln M} = \frac{(\ln M)^{1-p} - (\ln 2)^{1-p}}{1-p}.$$

which will exceed any finite number as M goes to infinity. So $\sum \frac{1}{n(\ln n)^p}$ also diverges in this case.

□

4. P120 Ex. 11

Proof. Since $a_{n+1} - a_n = \frac{1}{na_n}$ and $a_1 = 1$, the sequence $\{a_n\}$ is strictly increasing and $a_n \geq 1$ for all $n \in \mathbb{N}$.

We prove $\{a_n\}_{n \geq 1}$ diverges by contradiction. If the sequence $\{a_n\}_{n \geq 1}$ is bounded above by M , $0 < M < \infty$, then

$$1 \leq a_n \leq M \text{ for all } n.$$

So

$$\begin{aligned} a_1 &= 1, \\ a_2 - a_1 &= \frac{1}{na_1} \geq \frac{1}{M \times 1} \\ a_3 - a_2 &= \frac{1}{na_2} \geq \frac{1}{M \times 2} \\ &\dots \\ a_n - a_{n-1} &= \frac{1}{na_{n-1}} \geq \frac{1}{M \times (n-1)}. \end{aligned}$$

Hence

$$a_n \geq 1 + \frac{1}{M} \left(1 + \frac{1}{2} + \dots + \frac{1}{n-1} \right).$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the partial sum $s_n = \sum_{k=1}^n \frac{1}{k}$ converges to ∞ . This proves that $\{a_n\}_{n \geq 1}$ is not bounded. A contradiction. \square

5. P120 Ex. 12

Proof. $\{a_n\}$ converges.

We prove this by using the Monotone convergence theorem. Firstly $a_n \geq 1$ for all n and

$$a_{n+1} - a_n = \frac{1}{n^2 a_n} > 0.$$

So $\{a_n\}$ is increasing.

Secondly since $a_n \geq 1$ for all $n \geq 2$,

$$a_{n+1} - a_n = \frac{1}{n^2 a_n} \leq \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

In particular, for $n \geq 2$,

$$\begin{aligned}
 a_{n+1} - a_n &\leq \frac{1}{n-1} - \frac{1}{n}, \\
 a_n - a_{n-1} &\leq \frac{1}{n-2} - \frac{1}{n-1} \\
 a_{n-1} - a_{n-2} &\leq \frac{1}{n-3} - \frac{1}{n-2} \\
 &\dots\dots \\
 a_3 - a_2 &\leq \frac{1}{1} - \frac{1}{2}. \\
 a_2 &= 1 + 1 = 2.
 \end{aligned}$$

Adding all these together, we have,

$$a_{n+1} \leq 1 + 2 - \frac{1}{n} \leq 3.$$

This implies $\{a_n\}$ is bounded above. So it converges. □

6. P120 Ex. 13

Proof. We write out

$$\begin{aligned}
 d_{2^m} &= 1 + \frac{1}{2^p} + \dots + \frac{1}{2^{(m-1)p}} + \frac{1}{(2^{m-1} + 1)^p} + \dots + \frac{1}{2^{mp}} \\
 &= d_{2^{m-1}} + \frac{1}{(2^{m-1} + 1)^p} + \dots + \frac{1}{2^{mp}} \\
 &\leq d_{2^{m-1}} + \frac{1}{(2^{m-1} + 1)^p} \times 2^{m-1} \\
 &\leq d_{2^{m-1}} + \frac{1}{2^{(m-1)(p-1)}}.
 \end{aligned}$$

for all $m \geq 1$. Note that this is slightly different than the conclusion in **(a)**.

(b). For $m \geq 1$,

$$\begin{aligned}
 d_{2^m} - d_{2^{m-1}} &\leq \frac{1}{2^{(m-1)(p-1)}}, \\
 d_{2^{m-1}} - d_{2^{m-2}} &\leq \frac{1}{2^{(m-2)(p-1)}}, \\
 d_{2^{m-2}} - d_{2^{m-3}} &\leq \frac{1}{2^{(m-3)(p-1)}}, \\
 &\dots\dots \\
 d_3 - d_2 &\leq \frac{1}{2^{p-1}} \\
 d_2 - d_1 &\leq 1.
 \end{aligned}$$

So

$$d_{2^m} < 1 + 1 + \frac{1}{2^{p-1}} + \cdots + \frac{1}{2^{(m-1)(p-1)}} = 1 + \frac{1}{1 - (\frac{1}{2})^{p-1}}.$$

For any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$2^{m-1} \leq n \leq 2^m.$$

So

$$d_n \leq d_{2^m} \leq 1 + \frac{1}{1 - (\frac{1}{2})^{p-1}},$$

which implies that $\{d_n\}$ is bounded for $p > 1$; on the other hand $\{d_n\}$ is strictly increasing. So $\lim_{n \rightarrow \infty} d_n$ exists. Since d_n is the partial sum of $\sum \frac{1}{n^p}$, $\sum \frac{1}{n^p}$ converges. \square

7. P123. Ex. 1

a). $\sum \frac{n}{n+2}$ diverges as $\lim_{n \rightarrow \infty} a_n = 1$.

d). $\sum \frac{n!}{n^n}$ converges.

$$\frac{n!}{n^n} = \frac{1 \times 2 \times \cdots \times n}{n \times n \times n \times \cdots \times n} = \frac{1}{n^2} \frac{2}{n} \times \frac{3}{n} \times \frac{4}{n} \times \cdots \times \frac{n}{n} \leq \frac{2}{n^2}.$$

Since $\sum \frac{2}{n^2}$ converges, $\frac{n!}{n^n}$ converges.

g). $\sum \frac{n^2+1}{3n^3-n}$ diverges as

$$\frac{n^2+1}{3n^3-n} \geq \frac{n^2}{3n^3} = \frac{1}{3n}.$$

Since $\sum \frac{1}{n}$ diverges, so $\sum \frac{n^2+1}{3n^3-n}$ diverges.

j). $\sum \frac{3}{n^2+1}$ converges as

$$\sum \frac{3}{n^2+1} \leq \frac{3}{n^2}.$$

Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{3}{n^2+1}$ converges.

l). $\sum \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ converges as

$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{n(n+1)}} \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2n^{3/2}}.$$

Since $\sum \frac{1}{n^{3/2}}$ converges, the series $\sum \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ converges.

n). $\sum \left(\frac{1}{n} - \frac{1}{n+1} \right)$ converges as

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}.$$

8. P123. Ex. 2

Proof. We let $b_n = \sqrt[n]{a_n}$. Since $b_n \Rightarrow q$ and $0 \leq q < 1$, then by the definition of limits, for $\varepsilon = \frac{1-q}{2}$, there exists N such that for all $n \geq N$,

$$|b_n - q| < \varepsilon, \Rightarrow b_n < q + \varepsilon = q + \frac{1-q}{2} = \frac{1+q}{2} < 1.$$

$$a_n = b_n^n < \left(\frac{1+q}{2} \right)^n.$$

Since $\sum \left(\frac{1+q}{2} \right)^n$ converges, $\sum a_n$ converges.

By the reasoning above, for $\varepsilon = \frac{q-1}{2}$, there exists N such that for any $n \geq N$,

$$|b_n - q| < \varepsilon, \Rightarrow b_n > q - \varepsilon = \frac{1+q}{2} > 1.$$

Then

$$a_n = b_n^n \geq 1.$$

Since $a_n \geq 1$, $\sum a_n$ diverges. \square

9. P123. Ex. 3

Proof. Since $a_n \geq 0$ and $\sum a_n$ converges, we let $M = \sum a_n$. Then

$$M \geq 0 \text{ and } a_n \leq M \text{ for all } n.$$

Then

$$a_n^2 \leq M a_n.$$

Since $M \sum a_n$ converges, $\sum a_n^2$ converges. \square

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