HOMEWORK 1

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1. P 229. # 7.1.2.

Proof. **(a).** Let $f_n(x) = \frac{x^{99}+\frac{5}{n}}{x^{66}+\frac{x^3}{n}}$. Then

$$f_n \to x^{33}$$

as n goes to infinity. We estimate the difference,

$$f_n(x) - x^{33} = \frac{\frac{5}{n}}{x^{66} + \frac{3}{n}} \leq \frac{5}{nx^{66}} \leq \frac{5}{n},$$

for all $x \in [1, 3]$, which goes to zero as n goes to infinity. Thus for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$f_n(x) - x^{33} \leq \frac{5}{n} < \epsilon.$$

This proves that f_n converges to x^{33} uniformly on $[1, 3]$ as n goes to infinity. Therefore

$$\lim_{n\to\infty} \int_1^3 \frac{nx^{99} + 5}{x^3 + nx^{66}} dx = \int_1^3 x^{33} dx = \frac{3^{34} - 1}{34}.$$

(b). For $0 \leq x \leq 2$, $e^{\frac{x^2}{n}} \geq 1$; then we have

$$e^{\frac{x^2}{n}} - 1 = e^{\frac{x^2}{n}} - 1 \leq e^{\frac{4}{n}} - 1.$$

The sequence $\{e^{\frac{4}{n}} - 1\}$ goes to zero as n goes to infinity. By a similar proof in **part(a)**, $e^{\frac{x^2}{n}}$ goes to 1 uniformly as n goes to infinity. Therefore

$$\lim_{n\to\infty} \int_0^2 e^{x^2/n} dx = \int_0^2 e^0 dx = 2.$$

(c). For $0 \leq x \leq 3$,

$$\sqrt{\sin \frac{x}{n} + x + 1} - \sqrt{x+1} = \frac{|\sin \frac{x}{n}|}{\sqrt{\sin \frac{x}{n} + x + 1} + \sqrt{x+1}} \leq \sin \frac{3}{n},$$

if $n \geq 100$. The sequence $\frac{3}{n}$ goes to zero as n goes to infinity. By the proof in **part (a)**, $\sqrt{\sin \frac{x}{n} + x + 1}$ converges to zero uniformly on $[0, 3]$. Thus

$$\lim_{n\to\infty} \int_0^3 \left(\sqrt{\sin \frac{x}{n} + x + 1} dx = \int_0^3 \sqrt{x+1} dx = \frac{14}{3}.$$

□

2. P229. # 7.1.3.

Proof. If f_n converges to f uniformly on E, then by the uniform Cauchy criterion, for any $1 > \epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$,

$$|f_m(x) - f_n(x)| \leq \epsilon.$$

Let $m = N + 1$. Then

$$|f_n(x)| \leq |f_{N+1}(x)| + \epsilon \leq |f_{N+1}(x)| + 1.$$

Each f_n is bounded for $1 \leq n \leq N+1$. Thus $\{f_n\}$ is uniformly bounded on E.

On the other hand, to show that f is bounded, we use the uniform convergence of f_n to f. We omit the details. □

3. P230. # 7.1.6.

Proof. The sequence $\{f_n\}$ converges to f uniformly on E: for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon/3.$$

For f_N and for the same $\epsilon > 0$, there exists $\delta > 0$ such that for $|x - y| < \delta$,

$$|f_N(x) - f_N(y)| \leq \epsilon/3.$$

Therefore

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< 3 \times \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Thus f is uniformly continuous on E. □

4. P230. # 7.1.8.

Proof. Let $f_n(x) = (1+\frac{x}{n})^n e^{-x}$. For $x = a$, $f_n(a)$ converges to 1 as n goes to infinity. Each f_n is differentiable on (a, b). Next we show that $f_n'(x)$ converges uniformly on (a, b):

$$\begin{aligned} f_n'(x) &= (1+\frac{x}{n})^{n-1}e^{-x} - (1+\frac{x}{n})^n e^{-x} \\ &= (1+\frac{x}{n})^{n-1}(1-1-\frac{x}{n})e^{-x} \\ &= -\frac{x}{n}(1+\frac{x}{n})^{n-1}e^{-x} \\ &\leq \frac{x}{n} \leq \frac{b}{n}, \end{aligned}$$

since $(1+\frac{x}{n})^n$ is increasing and converges to e^x. Therefore $f_n'(x)$ converges uniformly on (a, b). By Theorem 7.12, f_n converges uniformly on (a, b). Thus

$$\lim_{n\to\infty} \int_a^b (1+\frac{x}{n})^n e^{-x} dx = \int_a^b 1 dx = b - a.$$

□

5. P230. # 7.1.10.

Proof. The sequence f_n converges to f uniformly on E: for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > N$,

$$|f_n(x) - f(x)| < \epsilon/2,$$

i.e., $-\epsilon/2 + f(x) < f_n(x) < f(x) + \epsilon/2$. We split the terms as follows.

$$\begin{aligned} &\frac{f_1(x) + f_2(x) + \cdots + f_n(x)}{n} - f(x) \\ &= \frac{f_1(x) + \cdots + f_{N-1}(x)}{n} + \frac{f_N(x) + \cdots + f_n(x) - nf(x)}{n} \\ &= g_n(x) + \frac{f_N(x) + \cdots + f_n(x) - nf(x)}{n}. \end{aligned}$$

This implies

$$g_n(x) - \frac{N}{n}f(x) + \frac{N\epsilon}{2n} - \epsilon/2 \leq \frac{f_1(x) + f_2(x) + \cdots + f_n(x)}{n} - f(x) \leq g_n(x) - \frac{N}{n}f(x) - \frac{N\epsilon}{2n} + \epsilon/2.$$

Since each f_n is bounded and f_n converges uniformly to $f(x)$, f is bounded. Moreover,

$$\lim_{n\to\infty} g_n(x) - \frac{N}{n}f(x) + \frac{N\epsilon}{2n} = 0 = \lim_{n\to\infty} g_n(x) - \frac{N}{n}f(x) - \frac{N\epsilon}{2n}.$$

Therefore for the same $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$-\frac{\epsilon}{2} < g_n(x) - \frac{N}{n}f(x) + \frac{N\epsilon}{2n} < \epsilon/2, -\epsilon/2 < g_n(x) - \frac{N}{n}f(x) - \frac{N\epsilon}{2n} < \epsilon/2.$$

We take $n = \max\{N, N_1\}$. For $n \geq N$, we have

$$-\epsilon < \frac{f_1(x) + f_2(x) + \cdots + f_n(x)}{n} - f(x) < \epsilon.$$

Therefore

$$\lim_{n\to\infty} \frac{f_1(x) + f_2(x) + \cdots + f_n(x)}{n} = f(x).$$

□

6. P230. # 7.1.11.

Proof. The sequence f_n converges to f uniformly on E and each f_n is integrable and hence is bounded; thus f is bounded by $M > 0$, i.e., $|f| < M$ for some $M > 0$. On the other hand, we have

$$\lim_{n\to\infty} b_n = 1,$$
$$\lim_{n\to\infty} f_n(x) = f(x), \text{ uniformly on E},$$
$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

The last one is implied by the second one. Note that b_n is increasing. So for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$

$$|b_n - 1| < \frac{\epsilon}{3M},$$
$$|f_m(x) - f(x)| < \frac{\epsilon}{3}, \text{ for all } x \in E,$$
$$\left| \int_0^1 f_n(x)dx - \int_0^1 f_m(x)dx \right| < \frac{\epsilon}{3}.$$

Taking $m = N$. Then

$$\begin{aligned} &\left| \int_0^{b_n} f_n(x)dx - \int_0^1 f(x)dx \right| \\ &= \left| \int_0^{b_n} f_n(x)dx - \int_0^{b_n} f_N(x)dx + \int_0^{b_n} f_N(x)dx - \int_0^1 f(x)dx \right| \\ &= \left| \int_0^{b_n} (f_n(x) - f_N(x))dx + \int_0^{b_n} (f_N(x) - f(x))dx + \int_{b_n}^1 f(x)dx \right| \\ &\leq \int_0^{b_n} |f_n(x) - f_N(x)|dx + \int_0^{b_n} |f_N(x) - f(x)|dx + M(1 - b_n) \\ &< \frac{\epsilon}{3} \times 3 = \epsilon. \end{aligned}$$

To conclude,

$$\lim_{n\to\infty}\int_0^{b_n} f_n(x)dx = \int_0^1 f(x)dx.$$

This completes the proof. ☐

7. P235. # 7.2.1

Proof. **(a).** Let x_0 be a point in the bounded interval in $\mathbb{R}$. Then for sufficiently large k,

$$\sin\frac{x_0}{k^2} \le \frac{|x_0|}{k^2}.$$

By the comparison test, the series $\sum_{k=1}^{\infty}\sin\frac{x}{k^2}$ converges absolutely and hence converges at x_0. On the other hand, let $f_k(x) = \sin\frac{x}{k^2}$,

$$f_k'(x) = \frac{1}{k^2}\cos\frac{x}{k^2}.$$

Therefore $|f_k'(x)| \le \frac{1}{k^2}$ for all x in the bounded interval. By the Weierstrass test, $\sum f_k'(x)$ converges uniformly. Thus $\sum_{k=1}^{\infty} f_k(x)$ converges.

(b). For all x in this closed subinterval of $(0,\infty)$,

$$e^{-kx} \le e^{-ka},$$

where a is the left endpoint of this interval. We claim $a > 0$. Indeed, the interval is contained in $(0,\infty)$ and so $a \ge 0$. If $a = 0$, the interval is open on the left endpoint. Thus $a > 0$.

Therefore by the Weierstrass test, $\sum_{k=1}^{\infty} e^{-kx}$ converges uniformly. ☐

8. P236. # 7.2.4.

Proof. Since $\frac{\cos kx}{k^2} \le \frac{1}{k^2}$, the series $\sum_{k=1}^{\infty}\frac{\cos kx}{k^2}$ converges uniformly on $\mathbb{R}$. Therefore

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} f(x)dx &= \sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\pi/2}\cos kx dx \\
&= \sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\frac{\pi}{2}}\cos kx dx \\
&= \sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\frac{\pi}{2}}\frac{1}{k}(\sin kx)'\,dx \\
&= \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^3}.
\end{aligned}$$

□

9. P236. # 7.2.7

Proof. Let $M > 0$ be the bound of g_1 on E and the partial sum $S_n(x)$ of $\sum_{k=1}^{\infty} f_k(x)$. Let $S(x)$ be the uniform limit of the series $\sum_{k=1}^{\infty} f_k(x)$. The series $f = \sum_{k=1}^{\infty} f_k(x)$ converges to $S(x)$ uniformly on E: for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > m > N$,

$$|S_n(x) - S_{m-1}(x)| \le \frac{\epsilon}{M},$$

and

$$|S_{m-1}(x) - S(x)| < \frac{\epsilon}{3M},$$

for all $x \in E$. Then

$$\begin{aligned}
&\sum_{k=m}^{n} \Big(f_k(x) g_k(x) \\
&= S_n(x)g_n(x) - S_{m-1}(x)h_m(x) + \sum_{k=m}^{n-1} \Big(S_k(x)\left(g_k(x) - g_{k+1}(x)\right) \\
&= (S_n(x) - S_{m-1}(x))g_n(x) + S_{m-1}(x)(g_n(x) - g_m(x)) + \sum_{k=m}^{n-1} \Big(S_k(x)\left(g_k(x) - g_{k+1}(x)\right) \\
&= |(S_n(x) - S_{m-1}(x))g_n(x) + (S_{m-1}(x) - S(x))(g_n(x) - g_m(x)) \\
&\qquad + \sum_{k=m}^{n-1} (-S)(g_k(x) - g_{k+1}(x)) + \sum_{k=m}^{n-1} \Big(S_k(x)\left(g_k(x) - g_{k+1}(x)\right) \\
&= |(S_n(x) - S_{m-1}(x))g_n(x) + (S_{m-1}(x) - S(x))(g_n(x) - g_m(x)) \\
&\qquad + \sum_{k=m}^{n-1} \Big(S_k(x) - S(x)\big)\left(g_k(x) - g_{k+1}(x)\right) \\
&< M|S_n(x) - S_{m-1}(x)| + \frac{\epsilon}{3M} \times M + \frac{\epsilon}{3M} \times M \\
&< M \times \frac{\epsilon}{3M} + \frac{\epsilon}{3M} \times M + \frac{\epsilon}{3M} \times M = \epsilon,
\end{aligned}$$

for all $x \in E$. By the Uniform Cauchy Criterion, $\sum_{k=1}^{\infty} f_k(x)g_k(x)$ converges uniformly on E. □

10. P236. # 7.2.10

Proof. Let $g_n(x) = \Big(\sum_{k=1}^{n} f_k(x))^{\frac{1}{n}}$. Since $0 \le f_k(x) \le f_{k+1}(x)$,

$$g_n(x) \Big\le (n f_n^n(x))^{\frac{1}{n}} = n^{\frac{1}{n}} f_n(x) \le 2 f_n(x)$$

for n large enough because $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. Since f_n converges to f uniformly on $[a,b]$, g_n converges uniformly on $[a,b]$. Therefore $\lim_{n\to\infty}\int_a^b g_n(x)dx$ exists and equals $\int_a^b \lim_{n\to\infty} g_n(x)dx$.

Since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$n^{\frac{1}{n}} < 1+\epsilon.$$

So $g_n(x) \leq (1+\epsilon) f_n(x)$ and hence

$$\int_a^b g_n(x)dx \leq (1+\epsilon)\int_a^b f_n(x)dx$$

Taking limits on both sides,

$$\int_a^b \lim_{n\to\infty} g_n(x)dx \leq (1+\epsilon)\int_a^b f(x)dx.$$

Since $\epsilon > 0$ is arbitrary,

$$\int_a^b \lim_{n\to\infty} g_n(x)dx \leq \int_a^b f(x)dx$$

Since $f_{k+1} \geq f_k(x) \geq 0$ on $[a,b]$,

$$\begin{aligned}
\sum_{k=1}^{n}\left(f_k^n(x)\right)^{\frac{1}{n}} &= \sum_{k=1}^{N-1} f_k(x) + \sum_{k=N}^{n}\left(f_k(x)\right)^{\frac{1}{n}} \\
&\geq \sum_{k=N}^{n}\left(f_N^n(x)\right)^{\frac{1}{n}} \\
&\geq (n-N)^{\frac{1}{n}} f_N(x).
\end{aligned}$$

Therefore taking $n \to \infty$,

$$\int_a^b \lim_{n\to\infty} g_n(x)dx \geq \int_a^b f_N(x)dx.$$

Taking $N \to \infty$,

$$\int_a^b \lim_{n\to\infty} g_n(x)dx \geq \int_a^b f(x)dx.$$

Therefore

$$\int_a^b \lim_{n\to\infty} g_n(x)dx = \int_a^b f(x)dx.$$

$\square$

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