HOMEWORK 10

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1. # 11.6.1

Proof. **(a).** The total derivative of f at (a,b) is

$$Df(a,b) = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}.$$

By the inverse function theorem,

$$D(f^{-1})(a,b) = \frac{1}{17}\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}.$$

(b). We compute (a,b) such that

$$\begin{cases} u+v & = 0, \\ \sin u + \cos v & = 1. \end{cases}$$

Then

$$u = \frac{k\pi}{2}, v = -\frac{k\pi}{2}, k \in \mathbb{Z}.$$

Then depending on where the pre-image of $(0,1)$ under f^{-1}, the total derivative of f has 4 possibilities.

$$Df(2n\pi, -2n\pi) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix};\text{ the inverse is } \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix};$$

and

$$Df(2n\pi + \pi/2, -2n\pi - \pi/2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};\text{ the inverse is } \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix};$$

and

$$Df(2n\pi + \pi, -2n\pi - \pi) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix};\text{ the inverse is } \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix};$$

and

$$Df(2n\pi + 3\pi/2, -2n\pi - 3\pi/2) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix};\text{ the inverse is } \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

□

2. # 11.6.2

Proof. **(a).** Let $F(x,y,z) = xyz + \sin(x+y+z)$. Then

$$\frac{\partial F}{\partial z}(0,0,0) = 1.$$

Then by the implicit function theorem, there exists an open set $W \subset \mathbb{R}^2$ containing $(0,0)$, and a unique continuously differentiable function g on W such that

$$z = g(t).$$

(b). Let $F(x,y,z) = x^2 + y^2 + z^2 + \sqrt{\sin(x^2+y^2) + 3z + 4} - 2$. Then

$$\frac{\partial F}{\partial z}(0,0,0) = \frac{3}{4}.$$

Then by the implicit function theorem, there exists an open set $W \subset \mathbb{R}^2$ containing $(0,0)$, and a uniquely differentiable function g on W such that

$$z = g(t).$$

□

3. # 11.6.3

Proof. Let $f(u,v,w,x,y) = (f_1, f_2, f_3)|_{(u,v,w,x,y)}$ such that

$$f_1 = u^5 + xv^2 - y + w, f_2 = v^5 = yu^2 - x + w, f_3 = w^4 + y^5 - x^4 - 1.$$

Then the partial Jacobian at $(1,1,-1,1,1)$ is

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{matrix} 4u^4 & 2xv & 1 \\ 2yu & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{matrix} = \begin{matrix} 4 & 2 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & -4 \end{matrix} = -64.$$

Therefore by the implicit function theorem, there exists $r > 0$ such that on $B_r(1,1)$ and a unique continuously differentiable function g such that $g(x,y) = (u(x,y), v(x,y), w(x,y))$ satisfying

$$g(1,1) = (1,1,-1).$$

□

4. # 11.6.7

Proof. By the implicit function theorem, for $j = 1, 2, \cdots, n$, there exists open sets W_j containing $(a_1, a_2, \cdots, a_n)$, an $r > 0$, and functions $g_j(u_j)$, $\mathbb{C}^1$ on W_j, such that

$$F(x_1, \cdots, x_{j-1}, g_j(u_j), x_{j+1}, x_{j+1}, \cdots, x_n) = 0 \text{ on } W_j.$$

We have n open sets $\{W_j\}_{1\le j\le n}$. The open set W_j contains $(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n)$. So there exists r_k^j, $1 \le k \le n$ and $k \ne j$ such that the Cartesian product

$$(a_1 - r_1^j, a_1 + r_1^j) \times \cdots \times (a_{j-1} - r_{j-1}^j, a_{j-1} + r_{j-1}^j) \times (a_{j+1} - r_{j+1}^j, a_{j+1} + r_{j+1}^j) \cdots \times (a_n - r_n^j, a_n + r_n^j)$$

is contained in the ball W_j. For a_j, we choose $r_j = \min\{a_j^k : 1 \le k \le n, k \ne j\}$. Then the Cartesian product

$$(a_1 - r_1, a_1 + r_1) \times \cdots (a_{j-1} - r_{j-1}) \times (a_{j+1} + r_{j+1}) \times \cdots \times (a_n - r_n, a_n + r_n)$$

contains $a = (a_1, \cdots, a_j, \cdots, a_n)$. Therefore there exists $B_r(a)$ with $r > 0$ such that if fixing a_j,

$$(x_1, \cdots, x_{j-1}, a_j, x_{j+1}, x_n) \in B_r(a), \Rightarrow (x_1, \cdots, x_{j-1}, x_{j+1}, x_n) \in W_j.$$

Since $\frac{\partial F}{\partial x_j}(a) \ne 0$, we decrease the size of r such that on $B_r(a)$,

$$\frac{\partial F}{\partial x_j}(x) \ne 0 \text{ for all } x \in B_r(a).$$

We work on $B_r(a)$. For $j = 1$, on W_1,

$$\frac{\partial F}{\partial x_n} + \frac{\partial F}{\partial x_1}\frac{\partial g}{\partial x_n} = 0.$$

Therefore

$$\frac{\partial g}{\partial x_n} = -\frac{\frac{\partial F}{\partial x_n}}{\frac{\partial F}{\partial x_1}}.$$

For $2 \le j \le n$,

$$\frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2}\frac{\partial g}{\partial x_1} = 0, \Rightarrow \frac{\partial g}{\partial x_1} = -\frac{\frac{\partial F}{\partial x_1}}{\frac{\partial F}{\partial x_2}}$$

$$\cdots, \cdots$$

$$\frac{\partial F}{\partial x_{j-1}} + \frac{\partial F}{\partial x_j}\frac{\partial g}{\partial x_{j-1}} = 0, \Rightarrow \frac{\partial g}{\partial x_{j-1}} = -\frac{\frac{\partial F}{\partial x_{j-1}}}{\frac{\partial F}{\partial x_j}}$$

$$\cdots, \cdots$$

$$\frac{\partial F}{\partial x_{n-1}} + \frac{\partial F}{\partial x_n}\frac{\partial g}{\partial x_{n-1}} = 0, \Rightarrow \frac{\partial g}{\partial x_{n-1}} = -\frac{\frac{\partial F}{\partial x_{n-1}}}{\frac{\partial F}{\partial x_n}}.$$

Multiplying them together, we obtain

$$\frac{\partial g_1}{\partial x_n}\frac{\partial g_2}{\partial x_1}\cdots\frac{\partial g_n}{\partial x_{n-1}} = (-1)^n$$

for all $x \in B_r(a)$. This proves the claim. □

5. # 11.6.10

Proof. **(a).** We compute

$$Df(t_0) = \begin{pmatrix} u'(t_0) \\ v'(t_0) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then either $u'(t_0) \neq 0$ or $v'(t_0) \neq 0$.

(b). Suppose that $u'(t_0) \neq 0$. Since u is twice continuously differentiable and $u(t_0) = x_0$, then by the inverse function theorem, there exists an open set W containing x_0 and a unique continuously differentiable function $g = u^{-1}$ on W such that

$$t = g(x), \text{ for all } x \in W.$$

Since $g = u^{-1}$ on W,

$$u(g(x)) = u \circ u^{-1}(x) = x.$$

□

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