HOMEWORK 2

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1. P247. # 7.3.2

Proof. **(a).** Let $a_k = 2^{-k}$. Then

$$\lim_{k\to\infty} \sqrt[k]{a_k} = 2^{-1}.$$

Thus $R = 2$. For $x = -2$,

$$\sum_{k=0}^{\infty} \frac{(-2)^k}{2^k} = \sum_{k=0}^{\infty} (-1)^k,$$

which diverges. The same claim holds for $x = 2$. Therefore the interval of convergence is $(-2, 2)$.

(b). Let $a_k = ((-1)^k + 3)^k$. Therefore

$$\limsup_{k\to\infty} \sqrt[k]{a_k} = \limsup_{k\to\infty} \Big((-1)^k + 3\Big) = 4.$$

This implies that

$$R = \frac{1}{4}.$$

For the two endpoints, $x = \frac{3}{4}$,

$$\sum_{k=1}^{\infty} \left(\frac{1 + 3(-1)^k}{4}\right)^k = \sum_{k:\text{even}} 1 + \sum_{k:\text{odd}} (-\frac{1}{2})^k,$$

which diverges. For $x = \frac{5}{4}$,

$$\sum_{k=1}^{\infty} \left(\frac{(-1)^k + 3}{4}\right)^k = \sum_{k:\text{even}} 1 + \sum_{k:\text{odd}} (\frac{1}{2})^k,$$

which diverges. Thus the interval of convergence is $(\frac{3}{4}, \frac{5}{4})$.

(c). Let $a_k = \log \frac{k+1}{k} = \log(1 + \frac{1}{k})$. Therefore

$$R = \lim_{k\to\infty} \frac{a_k}{a_{k+1}} = \lim_{k\to\infty} \frac{\log(1 + \frac{1}{k})}{\log(1 + \frac{1}{k+1})} = \lim_{k\to\infty} \frac{\frac{k}{k+1}}{\frac{k+1}{k+2}} = \lim_{k\to\infty} \frac{k(k+2)}{(k+1)^2} = 1.$$

For $x=-1$, the series $\sum_{k=1}^{\infty}(-1)^k\log(1+\frac{1}{k})$ converges by the alternating series test. For $x=1$, the series $\sum_{k=1}^{\infty}\log(1+\frac{1}{k})\geq\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. Indeed,

$$\log(1+x)\geq\frac{1}{2}x$$

for all $x\in[0,1]$. This can be proven by setting $f(x)=\log(1+x)-\frac{1}{2}x$ and verifying $f'(x)=\frac{1}{1+x}-\frac{1}{2}=\frac{1-x}{2(1+x)}\geq 0$. So the interval of convergence is $[-1,1)$.

(d). [1] Set $a_k=\frac{1\times 3\times\cdots\times(2k-1)}{(k+1)!}$. Then

$$\lim_{k\to\infty}\frac{a_k}{a_{k+1}}=\lim_{k\to\infty}\frac{k+2}{2k+1}=\frac{1}{2}.$$

So $|x^2|<\frac{1}{2}$. Thus $|x|<\frac{1}{\sqrt{2}}$.

At the endpoints, $x=\pm\frac{1}{\sqrt{2}}$, $x^2=\frac{1}{2}$. Therefore

$$\sum_{k=1}^{\infty}\frac{1\times 3\times\cdots\times(2k-1)}{(k+1)!}(\frac{1}{2})^k=\sum\frac{(2k)!}{2^{2k}k!(k+1)!}.$$

This converges because of the Stirling formula,

$$n!\sim n^n e^{-n}\sqrt{2\pi n},\ \text{for sufficiently large } n.$$

So the interval of convergence is $[\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}]$; the resulting comparison series is

$$\sum_{k=1}^{\infty}k^{-\frac{3}{2}}.$$

□

2. P247. # 7.3.5

Proof. Let $\{a_k\}$ be a sequence of bounded sequence of real numbers by $M>0$. Then for all $k\in\mathbb{N}$,

$$\limsup_{k\to\infty}\sqrt[k]{|a_k|}\leq\limsup_{k\to\infty}\sqrt[k]{M}=1.$$

Thus

$$R=\frac{1}{\limsup_{k\to\infty}\sqrt[k]{|a_k|}}\geq 1.$$

So this series has a positive radius of convergence, which might be positive infinity. □

[1] Thanks to Wang Xiaohua for pointing out a mistake in an earlier solution. Spring 2016.

3. P248. # 7.3.10

Proof. For $x \in [0,1]$, set

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^k a_k = \sum_{k=0}^{\infty} f_k(x) g_k(x),$$

where $g_k = a_k$ and $f_k(x) = (-1)^k x^k$. By Abel's theorem for uniform convergence, this series converges uniformly on $[0,1]$ and f is continuous on $[0,1]$. The latter implies that f is uniformly continuous continuous, for any $\epsilon > 0$, there is $\delta > 0$ such that $|x - y| < \delta$,

$$|f(x) - f(y)| < \epsilon.$$

Since the series converges to f,

$$\sum_{k=0}^{\infty} (-1)^k a_k (x^k - y^k) \ < \epsilon.$$

□

4. P259. # 7.4.1

Proof. **(a).** Let $f(x) = x^2 + \cos 2x$. Therefore

$$f \in \mathcal{C}^{\infty}(-\infty, \infty).$$

For any $C > 0$ and $n = 1$,

$$|f'(x)| \leq 2C + 2.$$

and $n \geq 2$,

$$|f'(x)| \leq 3^n.$$

By Theorem 7.4.3, f is analytic on $(-C, C)$. Since $C > 0$ is arbitrary, f is analytic on $\mathbb{R}$. To find the Maclaurin expansion,

$$f(0) = 1, \quad f'(0) = 0, f''(0) = -2.$$

Since the derivatives of f involve $\sin 2x$ and $\cos 2x$, at $x = 0$, $\sin 2x = 0$ and $\cos 2x = 1$. Therefore for all $n \geq 2$,

$$f^{2n}(0) = (-1)^n 2^{2n}.$$

Therefore the Maclaurin series

$$f(x) = 1 - x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n}}{n!} x^{2n}.$$

(c). Let $f(x) = \cos^2 x - \sin^2 x = \cos 2x$. Since the derivatives of f involve $\sin 2x$ and $\cos 2x$, at $x = 0$, $\sin 2x = 0$ and $\cos 2x = 1$. Therefore for all $n \geq 1$,

$$f^{2n}(0) = (-1)^n 2^{2n}, f^{2n+1}(0) = 0.$$

So f is analytic on $\mathbb{R}$ by Theorem 7.4.3. The Maclaurin series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{n!} x^{2n}.$$

□

5. P260. # 7.4.4

Proof. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. By Theorem 7.39,

$$f \in \mathcal{C}^{\infty}(-\infty, \infty);$$

and

$$a_k = \frac{f^{(k)}(0)}{k!}, \text{ for } 1 \leq k \leq n.$$

For $k \geq n + 1$,

$$a_k = 0.$$

□

6. P260. # 7.4.6

Proof. By Theorem 7.50 and the change of variables,

$$\lim_{n \to \infty} R_n(x) = 0,$$

for all $x \in \mathbb{R}$. Therefore the Taylor series converges to f for all $x \in \mathbb{R}$, and

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k, \text{ for all } x \in \mathbb{R}.$$

By definition of analytic functions, f is analytic on $(-\infty, \infty)$. □

7. P260. # 7.4.8

Proof. Suppose that f is analytic on (a, b). Therefore for all $x_0 \in (a, b)$, there is (c, d) such that $x_0 \in (c, d)$ and $(c, d) \subset (a, b)$ and f has a series expansion,

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \text{ for all } x \in (c, d).$$

By Theorem 7.30,

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}$$

for all $x \in (c, d)$. This infinite series converges to f; therefore f' is analytic.

Conversely, suppose that f' is analytic on (a, b). For all $x_0 \in (a, b)$, there is (c, d) such that $x_0 \in (c, d)$ and $(c, d) \subset (a, b)$, and f' has a series expansion,

$$f'(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k, \text{ for all } x \in (c, d).$$

There exists a closed interval $[c_1, d_1] \subset (c, d) \subset (a, b)$ such that

$$f'(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k, \text{ for all } x \in [c_1, d_1].$$

By Abel's Theorem, f' is continuous and the series converges uniformly on $[c_1, d_1]$. By the fundamental theorem of calculus, for all $x \in [c_1, d_1]$,

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t) dt = \sum_{k=0}^{\infty} b_k \int_{x_0}^{x} (t - x_0)^k dt = \sum_{k=0}^{\infty} \frac{b_k}{k+1} (x - x_0)^{k+1},$$

which implies

$$f(x) = f(x_0) + \sum_{k=0}^{\infty} \frac{b_k}{k+1} (x - x_0)^{k+1}.$$

Therefore f is analytic on (c_1, d_1).

□

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