HOMEWORK 3

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1. P 278. # 8.1.2

Proof. **a).**By Cauchy-Schwarz,

$$\|(a \cdot b)c\| = |a \cdot b|\|c\| \leq \|a\|\|b\|\|c\| \leq 1.$$

Therefore

$$\|v\| \leq \frac{1+1+1}{3} = 1.$$

Thus $v \in \mathbf{B}$.

b).

$$|a \cdot c - b \cdot d| = |(a-d) \cdot c + (c-b) \cdot d| \leq \|a-d\| + \|b-c\|.$$

c). By Cauchy-Schwarz, (iv) and (vi) in Theorem 8.9,

$$|a \cdot (b \times c)| = |(a \times b) \times c| \leq \|a \times b\|\|c\| \leq \|a \times b\|.$$

Thus

$$\|a \times b\|^2 = (a \times a)(b \times b) - (a \times b)^2.$$

Therefore

$$\sqrt{|a \cdot (b \times c)|^2 + |a \times b|^2} \leq \sqrt{\|a\|^2\|b\|^2} \leq 1.$$

□

2. P 280. # 8.1.9

Proof. For each k,

$$|a_k b_k| \leq \frac{a_k^2 + b_k^2}{2}.$$

Since $\sum_k a_k^2$ and $\sum b_k^2$ converges, $\sum_k a_k b_k$ converges absolutely. □

3. P286. # 8.2.4

Proof. **a).**

$$T(x,y,z,w) = (0, x+y, x-z, x+y+w) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

b).

$$T(x,y,z,w) = x - y + z = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix}.$$

c).

$$T(x_1, \cdots, x_n) = (x_1 - x_2, \cdots, x_n - x_1) = \begin{pmatrix} 1 & 0 & \cdots & -1 \\ -1 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

□

4. P287. # 8.2.11

Proof. **a).** For all $x \in \mathbb{R}^n$ with $\|x\| = 1$,

$$\|Tx\| \leq \|T\|\|x\| = \|T\|.$$

Therefore

$$M_1 \leq \|T\|.$$

b). Since T is linear, for any $\alpha \in \mathbb{R}$,

$$\alpha T(x) = T(\alpha x).$$

Thus for $x \neq 0$,

$$\frac{\|T(x)\|}{\|x\|} = T(\frac{x}{\|x\|}) \leq M_1$$

since $\|\frac{x}{\|x\|}\| = 1$.

c). By the definition of operator norm,

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} \leq M_1.$$

From (a),

$$M_1 \leq \|T\|.$$

So $M_1 = \|T\|$.

To prove that $M_2 = \|T\|$, on the one hand, for any $C > 0$ satisfying $\|T(x)\| \leq C\|x\|$,

$$\|T\| \leq C.$$

Taking inf on the right hand,

$$\|T\| \leq M_2.$$

On the other hand, By Theorem 8.17, $\|T\|$ is finite and

$$\|T(x)\| \leq \|T\|\|x\|.$$

Therefore

$$M_2 \leq \|T\|.$$

So $\|T\| = M_2$. □

5. P296. # 8.3.4

Proof. We skip graphing U. We know that E_2 is an open set in $\mathbb{R}^2$. Therefore

$$U = E_1 \cap E_2,$$

Therefore U is relatively open in E_1. On the other hand, since E_1 is closed in $\mathbb{R}^2$, U is relatively closed in $\mathbb{R}^2$. □

6. P296. # 8.3.6

Proof. **a).** C is relatively closed in E: there exists a closed set $B \in \mathbb{R}^n$ such that

$$C = E \cap B.$$

Since E is also closed, C is closed. On the other hand, since C is closed and $C \subset E$,

$$C = C \cap E.$$

Thus C is relatively closed in E.

b). C is relatively closed in E: there exists a closed set $B \in \mathbb{R}^n$,

$$C = E \cap B.$$

So

$$E \setminus C = E \cap (C)^c = E \cap (E \cap B)^c = E \cap (E^c \cup B^c) = E \cap B^c.$$

Since B^c is open, $E \setminus C$ is relatively open in E. □

7. P296. # 8.3.7

Proof. **a).** We prove it by contradiction. Suppose $A \cup B$ is not connected. There exist U and V, relatively open in $A \cup B$ such that

$$U \cap V = \emptyset, U \cup V = A \cup B.$$

We divide the discussion into 3 cases.

Case 1. $A \cap U \neq \emptyset$ and $B \cap V \neq \emptyset$. We consider $U_1 = A \cap U$ and $V_1 = A \cap V$. We show that A is separated by U_1 and V_1. Firstly

$$U_1 \cap V_1 = (A \cap U) \cap (A \cap V) = A \cap (U \cap V) = \emptyset,$$

since $U \cap V = \emptyset$.

Secondly, U is relatively open in $A \cup B$: there exists an open set G in $\mathbb{R}^n$ such that $U_1 = A \cap U = A \cap ((A \cup B) \cap G) = A \cap (A \cap G) = A \cap G$. Thus U_1 is relatively open in A; similarly V_1 is relatively open in A.

Thirdly $U_1 \cup V_1 = (A \cap U) \cup (A \cap V) = A \cap (U \cup V) = A \cap (A \cup B) = A$. Therefore A is separately by U_1 and V_1. Therefore A is not connected. A contradicition.

Case 2. We may assume that $A \cap U = \emptyset$. Since $A \cup B = U \cup V$, then intersecting both sides by U, $(U \cap A) \cup (U \cap B) = U$. Therefore $U \cap B = U$. Therefore U is a subset of B. We consider $B_1 = B \cap V$ and $B_2 = B \cap U$. Therefore firstly

$$B_1 \cup B_2 = B \cap (U \cap V) = B.$$

Secondly by the proof of **Case 1**, B_1 and B_2 are relatively open sets in V. Since $U = B \cap U$ is not empty, $B_1 \neq \emptyset$. If $B_2 = B \cap V \neq \emptyset$, we see that B is separately by B_1 and B_2. This is a contradiction to that B is connected. The proof that $B_2 = \emptyset$ is deferred to the third section.

Case 3. $B \cap V = \emptyset$ under the condition that $A \cap U = \emptyset$. Because $A \cup B = U \cup V$, intersecting both sides by A, then

$$A = A \cap V.$$

Thus

$$A \subset V \subset B^c.$$

Therefore

$$A \cap B = \emptyset.$$

This is a contradiction to that $A \cap B \neq \emptyset$.

b). Suppose that $E = \cup_\alpha E_\alpha$ is not connected. Then by the remark or Theorem 8.38 following Remark 8.29, there exists a pair of open sets A and

B in $\mathbb{R}^n$ such that

$$E \cap A \neq \emptyset; E \cap B \neq \emptyset, E \subset A \cup B, A \cap B = \emptyset.$$

By the first two conditions, there exist $\alpha_0 \in A$ and $\beta_0 \in B$ such that

$$E_{\alpha_0} \cap A \neq \emptyset, E_{\beta_0} \cap B \neq \emptyset.$$

Let $E_0 = E_{\alpha_0} \cup E_{\beta_0}$. Then

$$E_0 \cap A \neq \emptyset, E_0 \cap B \neq \emptyset.$$

Since $E \subset A \cup B$,

$$E_0 \subset A \cup B.$$

So by Remark 8.29, E_0 is not connected. On the other hand, $E_{\alpha_0} \cap E_{\beta_0} \neq \emptyset$, and $E_{\alpha_0}, E_{\beta_0}$ are connected sets, then part **a)**, E_0 is connected. A contradiction. Therefore E is connected.

c). If A and B are connected in $\mathbb{R}$, then by Theorem 8.30, A and B are intervals: suppose that

$$A = [a_1, b_1], \quad B = [a_2, b_2].$$

Suppose that A is to the left of B. Then since $A \cap B \neq \emptyset$,

$$A \cap B = [a_2, b_1].$$

This is an interval; by Theorem 8.30, $A \cap B$ is an interval.

d). Let $A := \{(x, y) : -1 \leq x \leq 1, -|x| \leq y \leq |x|\}$, and $B := \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\} \setminus \{0\}$. Then A and B are connected sets by a similar proof as in Remark 9.34. However

$$A \cap B = \{(x, y) : -1 \leq x \leq 1, -|x| \leq y \leq |x|\} \setminus \{0\}.$$

This set is not connected by using Remark 8.29, $A = \{(x, y) : x < 0\}$ and $B = \{(x, y) : x > 0\}$.

□

8. P296. # 8.3.8

Proof. **a).** Let A be an index set. If B_α is open for each $\alpha \in A$, then $\cup_\alpha B_\alpha$ is open. On the other hand, if V is open, for each $x \in V$, there exists $r > 0$ such that $B_r(x) \subset V$. Then

$$\cup_{x \in V} B_\alpha = V.$$

Here the index set is V.

b). Let $B_n = [\frac{1}{n}, 1 - \frac{1}{n}]$. Then

$$\cup_{n \geq 2} B_n = (0, 1).$$

So the conclusion in part **a)** fails if "open" is replaced by "closed." $\square$

9. P296. # 8.3.9

Proof. We know that E is closed; E^c is open. For each $a \in E^c$, there exists $r > 0$,

$$B_r(a) \in E^c.$$

So for any $x \in E$, $\|x - a\| \geq r > 0$ since x is not in the ball $B_r(a)$. Therefore

$$\inf_{x \in E} \|x - a\| \geq r > 0.$$

$\square$

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