

HOMEWORK 4

SHUANGLIN SHAO

1. P 301 # 8.4.2

Proof. We skip the graph-sketching. □

2. P301. # 8.4.4

Proof. a). Let

$$B = \cup\{V : V \subset A \text{ and } V \text{ is relatively open in } E\}.$$

This collection is not empty since \emptyset is one choice. It is not hard to see that B is the largest subset of A , which is relatively open set in E .

b). Let

$$B = \cap\{C : C \subset A \text{ and } C \text{ is relatively closed in } E \}.$$

This collection is not empty since $\bar{A} \cap \bar{E}$ is the smallest set containing A , which is relatively closed in E . □

3. P302. # 8.4.6

Proof. Use the theorem that any connected set E in \mathbb{R} is an interval, and the interval in \mathbb{R} is in the following 4 forms:

$$(a, b), [a, b], (a, b], [a, b]$$

where a, b are extended real numbers. The interior of E is, in any case,

$$(a, b)$$

which is connected. □

4. P302. # 8.4.9

Proof. In the three cases, we choose

$$A = (0, 1], B = (1, 2).$$

□

5. P348. # 10.1.2

Proof. The sequence $\{x_k\}$ is bounded in X : there exists $M > 0$ such that

$$\rho(x_n, b) < M$$

for some $b \in X$. For any $a \in X$,

$$\rho(x_n, a) < \rho(x_n, b) + \rho(b, a) < M + \rho(b, a) < \infty.$$

The converse implication is clear.

□

6. P 349. # 10.4.9

Proof. Suppose that $\{x_n\}$ and $\{y_n\}$ converge to $a \in X$. Then by the triangle inequality

$$\rho(x_n, y_n) \leq \rho(x_n, a) + \rho(y_n, a).$$

If sending n to infinity, then

$$\rho(x_n, y_n) \rightarrow 0,$$

as $n \rightarrow \infty$.

In the Euclidean space $(X, |\cdot|)$, we choose that $x_n = y_n = n$.

□

7. P349. # 10.1.7

Proof. Let $\{x_n\}$ be a Cauchy sequence in the discrete space. Then for $\epsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$,

$$\rho(x_n, y_n) < \frac{1}{2}.$$

By the definition in the discrete space,

$$x_n = y_n$$

for all $n \geq N$. This proves that, after finitely many terms,

$$x_n = a,$$

which proves that $\{x_n\}$ converges to a as n goes to infinity. Therefore the discrete space is complete. \square

8. P 349. # 10.1.8

Proof. a). The space $\mathcal{C}[a, b]$ is discussed in the example 10.6. By Lemma 7.11, any Cauchy sequence $\{f_n\}$ converges uniformly to f on E . By uniform convergence,

$$f \in \mathcal{C}[a, b].$$

b). We denote the distance function by ρ . Then to prove ρ is a metric on $\mathcal{C}[a, b]$, here we only verify that

$$\rho(f, g) = 0 \text{ implies that } f(x) = g(x) \text{ on } [a, b].$$

The integral $\int_a^b |f(x) - g(x)| dx = 0$ and $f - g$ is continuous on $[a, b]$; then

$$|f(x) - g(x)| = 0$$

for all $x \in [a, b]$. Therefore $f(x) = g(x)$.

c). Let $f_n(x) = x^n$ and $[a, b] = [0, 1]$. Then it is not hard to see that f_n is a Cauchy sequence on $[0, 1]$.

If $\mathcal{C}[a, b]$ is complete in the metric ρ . Then $\{f_n\}$ converges to some function, which belongs to $\mathcal{C}[a, b]$.

Let

$$g(x) = \begin{cases} = 0, & 0 \leq x < 1, \\ = 1, & x = 1. \end{cases}$$

Then $\rho(f_n, g) \rightarrow 0$ as n goes to infinity. We know that $L^1([a, b])$ is complete in the metric ρ . Then by Theorem 10.21, $\mathcal{C}[a, b]$ is a closed subset of $L^1([a, b])$ (**some knowledge of the Lebesgue space, $L^p, 1 \leq p < \infty$, is needed.**). Then by the equivalent characterization of closed sets in Theorem 10.16, then

$$g \in \mathcal{C}[a, b].$$

However g has a discontinuity point at $x = 1$. \square

9. P 349. # 10.1.10

Proof. a). Suppose that E is sequentially compact. Let $\{x_n\} \in E$ and

$$a = \lim_{n \rightarrow \infty} x_n.$$

We just need to prove that $a \in E$. By the sequentially compactness of E , there is a subsequence $\{x_{n_k}\}$ converges to some point in E . Then they converges to the same limit. Thus

$$a \in E.$$

This proves that E is closed.

To prove that E is bounded. If not, there is $\{x_n\} \in E$ such that $\rho(x_n, a) < n$. Since E is sequentially compact, there is $\{x_{n_k}\}$ converging to x_0 in E . Then there exists $K \in \mathbb{N}$ such that

$$\rho(x_{n_k}, x_0) < \frac{1}{2}.$$

Then by the triangle inequality,

$$\rho(x_{n_k}, x_0) > \rho(x_{n_k}, a) - \rho(a, x_0) > n_k - \rho(a, x_0).$$

Let $k \rightarrow \infty$, $n_k \rightarrow \infty$. Then

$$\rho(x_{n_k}, x_0) \rightarrow \infty.$$

This is a contradiction. Therefore E is bounded.

b). \mathbb{R} is closed but \mathbb{R} is not sequentially compact because $\{x_n = n\}$. This sequence does not converge in \mathbb{R} .

c). Let E be a bounded and closed subset of \mathbb{R} . Suppose $\{x_n\} \in E$, $\{x_n\}$ is bounded. By the Bolzano-Weierstrass theorem, $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$, which converges to $x_0 \in \mathbb{R}$. Since E is closed and $x_{n_k} \in E$ and $\lim_{k \rightarrow \infty} x_{n_k} = x_0$,

$$x_0 \in E.$$

Therefore E is sequentially compact.

□

DEPARTMENT OF MATHEMATICS, KU, LAWRENCE, KS 66045

E-mail address: slshao@math.ku.edu